

# MINIMAL CHARACTERISTIC BISETS FOR FUSION SYSTEMS

MATTHEW GELVIN AND SUNE PRECHT REEH

**ABSTRACT.** We show that every saturated fusion system  $\mathcal{F}$  has a unique minimal  $\mathcal{F}$ -characteristic biset  $\Lambda_{\mathcal{F}}$ . We examine the relationship of  $\Lambda_{\mathcal{F}}$  with other concepts in  $p$ -local finite group theory: In the case of a constrained fusion system, the model for the fusion system is the minimal  $\mathcal{F}$ -characteristic biset, and more generally, any centric linking system can be identified with the  $\mathcal{F}$ -centric part of  $\Lambda_{\mathcal{F}}$  as bisets. We explore the grouplike properties of  $\Lambda_{\mathcal{F}}$ , and conjecture an identification of normalizer subsystems of  $\mathcal{F}$  with subbisets of  $\Lambda_{\mathcal{F}}$ .

## 1. INTRODUCTION

If  $S$  is a Sylow  $p$ -subgroup of a finite group  $G$ , we talk about the fusion system  $\mathcal{F}_S(G)$  as an organizational framework for understanding the  $p$ -local structure of  $G$ . The fusion data is encoded as a category: The objects of  $\mathcal{F}_S(G)$  are the subgroups of  $S$ , and the morphisms are the maps between subgroups induced by conjugation in  $G$ . More generally, Puig introduced the notion of an abstract fusion system on  $S$ : This is again a category  $\mathcal{F}$  with objects the subgroups of  $S$  and morphism certain injective group maps between subgroups (see Section 2).

An abstract fusion system does not necessarily arise from a group in this manner, but we still think of the morphisms in  $\mathcal{F}$  as given by the conjugation action of some grouplike object on the subgroups of  $S$ . The notion of a characteristic biset turns this perspective around, and considers how  $S$  acts on the object that does the conjugating.

For  $S \in \text{Syl}_p(G)$  and the fusion system  $\mathcal{F}_S(G)$  realized by  $G$ 's conjugation action on  $S$ , we can ask how  $S$  acts on  $G$  by left and right multiplication. That is, we consider the  $(S, S)$ -biset  ${}_S G_S$ . For  $g \in G$ , if  $(b, a) \in S \times S$  is such that  $b \cdot g = g \cdot a$ , then  $b = {}^g a$ . In other words, fusion data ( $b = {}^g a$ ) is encoded in the biset structure ( $b \cdot g = g \cdot a$ ). This justifies calling  ${}_S G_S$  a *characteristic biset* for  $\mathcal{F}_S(G)$ .

Linckelmann and Webb extracted the features of  ${}_S G_S$  that are essential for understanding the fusion system  $\mathcal{F}_S(G)$ , resulting in a notion of characteristic bisets for any abstract fusion system  $\mathcal{F}$ . Fix a  $p$ -group  $S$ , a fusion system  $\mathcal{F}$  on  $S$ , and an  $(S, S)$ -biset  $\Omega$ .  $\Omega$  is then a characteristic biset for  $\mathcal{F}$  if:

- (0)  $\Omega$  is free both as a left and right  $S$ -set.

This implies that any  $\omega \in \Omega$  has stabilizer  $\{(b, a) \in S \times S \mid b \cdot \omega = \omega \cdot a\}$  of the form  $(P, \varphi) := \{(\varphi(a), a) \mid a \in P\}$  for  $P$  a subgroup of  $S$  and  $\varphi: P \hookrightarrow S$  some group injection.

Heuristically, this says that  $\omega$  “conjugates”  $a$  to  $\varphi(a)$ .

- (1) If  $\omega \in \Omega$  has stabilizer  $(P, \varphi)$ , then  $\varphi$  is a morphism of  $\mathcal{F}$ .

This means that all the conjugation induced by  $\Omega$  is in  $\mathcal{F}$ .

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- (2) For subgroups  $P$  of  $S$ ,  $\mathcal{F}$ -morphisms  $\varphi: P \rightarrow S$ , and  $\mathcal{F}$ -isomorphisms  $\eta_1: Q \xrightarrow{\cong} P$ ,  $\eta_2: \varphi P \xrightarrow{\cong} R$ , there is an equality of fixed-point set orders:  $|\Omega^{(P, \varphi)}| = |\Omega^{(Q, \eta_2 \varphi \eta_1)}|$ .

This condition generalizes the fact that, if  $G$  acts on a set  $X$ , then conjugate subgroups of  $G$  have fixed-point sets of equal size.

- (3)  $|\Omega|/|S|$  is prime to  $p$ .

This Sylow condition generalizes  $S \in \text{Syl}_p(G)$ .

The connection between a fusion system  $\mathcal{F}$  and an associated  $\mathcal{F}$ -characteristic biset is very strong:

- If  $\mathcal{F}$  is *saturated* (i.e., it satisfies the axioms needed to make  $\mathcal{F}$  look like the fusion induced by a finite group), then there exists a characteristic biset ([BLO]).
- If a characteristic biset for  $\mathcal{F}$  exists, then  $\mathcal{F}$  is saturated ([Pui2], also see [RS1] for a  $p$ -localized version).
- As suggested by Axioms (1) and (2), the characteristic biset determines  $\mathcal{F}$ .

If we allowed ourselves to think about virtual bisets in the double Burnside ring of  $S$ , and  $p$ -localized, then the converse to the last point would be true: Every saturated fusion system determines and is determined by a unique *characteristic idempotent* in  $A(S, S)_{(p)}$ , see [Rag]. However, motivated by Park's Theorem that an  $\mathcal{F}$ -characteristic biset gives rise to ambient, finite (but not necessarily Sylow) supergroup realizing  $\mathcal{F}$  ([Par1]) and subsequent work investigating smallest  $\mathcal{F}$ -characteristic biset orders in certain examples ([Par2]), we will opt to instead remain in the world of honest bisets.

For us then, the uniqueness of  $\mathcal{F}$ -characteristic bisets always fails: If  $G$  and  $H$  both contain  $S$  as a Sylow  $p$ -subgroup, and if  $\mathcal{F}_S(G) = \mathcal{F}_S(H) =: \mathcal{F}$ , then both  ${}_S G_S$  and  ${}_S H_S$  are characteristic bisets for  $\mathcal{F}$ , but they need not be equal (e.g.,  $H = G \times K$  for  $K$  your favorite  $p'$ -group). While a characteristic biset determines the fusion system, the fusion system does not determine the characteristic biset.

This paper proposes to solve this indeterminacy problem: In Theorem 5.3 we give a complete parameterization of all  $\mathcal{F}$ -characteristic bisets, which in particular implies

**Theorem A** (Corollary 5.4). *Every saturated fusion system  $\mathcal{F}$  has a unique minimal characteristic biset  $\Lambda_{\mathcal{F}}$ .*

Here, minimality means that if  $\Omega$  is any  $\mathcal{F}$ -characteristic biset, then  $\Lambda_{\mathcal{F}} \subseteq \Omega$  as  $(S, S)$ -biset. This makes  $\Lambda_{\mathcal{F}}$  the most natural choice of  $\mathcal{F}$ -characteristic biset, and we argue that it should be thought of as *the* characteristic biset by proving several additional Theorems B-E justifying this choice.

The preliminary Sections 2 and 3 contain the necessary background material for this paper. Corollary 3.8 in particular will play an essential role in identifying  $\Lambda_{\mathcal{F}}$ .

Section 4 contains the main technical background relating  $S$ -sets to  $\mathcal{F}$ -fusion needed in our search for  $\Lambda_{\mathcal{F}}$ . If  $\mathcal{F}$  is a saturated fusion system on  $S$ , and  $X$  is a finite  $S$ -set, we say that  $X$  is  $\mathcal{F}$ -stable if for all  $\mathcal{F}$ -morphisms  $\varphi: P \rightarrow S$ , we have an equality of fixed-point set orders  $|X^P| = |X^{\varphi P}|$  (cf. Axiom (2) for characteristic bisets). Just as the transitive  $G$ -sets form the basis for commutative monoid of all finite  $G$ -sets, the first author conjectured that the commutative monoid of  $\mathcal{F}$ -stable  $S$ -sets is free with basis naturally corresponding to the  $\mathcal{F}$ -conjugacy classes of subgroups of  $S$ . The second author proved this in [Ree]. We recall the defining features of these elements in Theorem 4.5, and provide a new proof that they actually form a basis in Corollary 4.7.

In Section 5 we prove Theorem A by rephrasing the problem as looking for a particular kind of  $\mathcal{F} \times \mathcal{F}$ -stable  $S \times S$ -set.

It should be emphasized that the parameterization of  $\mathcal{F}$ -characteristic bisets, and hence the construction of  $\Lambda_{\mathcal{F}}$ , relies solely on a straightforward counting argument, inductively indexed on the objects of  $\mathcal{F}$ .  $\Lambda_{\mathcal{F}}$  is therefore much easier to get a hold of than most of the objects that appear in  $p$ -local finite group theory. Even so, it turns out that there are deep connections between the minimal  $\mathcal{F}$ -characteristic biset and other, more complicated structures. We take this as further evidence for the special role played by  $\Lambda_{\mathcal{F}}$ , and devote the rest of the paper to exploring these connections.

Section 6 examines the minimal characteristic bisets of constrained fusion systems, which we view as the building blocks from which all fusion systems are glued together. If  $O_p(\mathcal{F})$  denotes the maximal normal subgroup of  $\mathcal{F}$  (so that every morphism of  $\mathcal{F}$  extends to induce an automorphism of  $O_p(\mathcal{F})$ ), we say that  $\mathcal{F}$  is *constrained* if  $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$ . Constrained fusion systems always come from finite groups, and in fact among all finite groups inducing such an  $\mathcal{F}$  there is a well defined minimal example. This finite group  $M_{\mathcal{F}}$  is the *model* of  $\mathcal{F}$ , which is characterized by requiring that  $C_{M_{\mathcal{F}}}(O_p(M_{\mathcal{F}})) \leq O_p(M_{\mathcal{F}})$ . As the constrained fusion system  $\mathcal{F}$  has both a minimal characteristic biset and a minimal group inducing  $\mathcal{F}$ , we might ask about the relationship between the two.

**Theorem B** (Theorem 6.7). *If  $\mathcal{F}$  is a constrained fusion system with minimal characteristic biset  $\Lambda_{\mathcal{F}}$  and model  $M_{\mathcal{F}}$ , then  ${}_S(M_{\mathcal{F}})_S = \Lambda_{\mathcal{F}}$  as  $(S, S)$ -bisets.*

In Section 7 we turn to more general fusion systems. If  $\mathcal{F}$  is not constrained, then there is no particularly good notion of a “minimal” group inducing  $\mathcal{F}$ ; indeed, in the case of exotic fusion systems there may be no finite Sylow supergroup at all. Even in these cases we can still talk about an associated  $p$ -local finite group, which is formed by augmenting the fusion system with an auxiliary category  $\mathcal{L}$ , the *centric linking system*. The morphisms of  $\mathcal{L}$  represent group elements whose conjugation actions induce the morphisms of  $\mathcal{F}$ ; this is made precise in Chermak’s notion of a *partial group* (of which  $\mathcal{L}$  is the motivating example), which is effectively a different method of packaging the data of a linking system.

In [Che] it was shown that every saturated fusion system has a unique associated centric linking system, using the Classification Theorem of Finite Simple Groups. Independent of this result, if we assume that a linking system  $\mathcal{L}$  exists, the axioms governing its structure allow us to define an  $(S, S)$ -biset structure on the set  $\mathfrak{I}$  of nonextendable isomorphisms of  $\mathcal{L}$ . While  $\mathfrak{I}$  is not  $\Lambda_{\mathcal{F}}$ , we do have  $\mathfrak{I} \subseteq \Lambda_{\mathcal{F}}$  as  $(S, S)$ -bisets. Moreover, we can identify  $\mathfrak{I}$  as the elements of  $\mathcal{L}$  that conjugate an object of  $\mathcal{L}$  (an  $\mathcal{F}$ -centric subgroup) into  $S$ :

**Theorem C** (Theorem 7.9). *If  $\mathcal{L}$  is a centric linking system associated to  $\mathcal{F}$ , then the  $(S, S)$ -biset of nonextendable isomorphisms  $\mathfrak{I}$  is the  $\mathcal{F}$ -centric part of  $\Lambda_{\mathcal{F}}$ .*

It should be noted that this biset  $\mathfrak{I}$  is just the elements of the partial group  $\mathcal{L}$ .

We interpret this result as saying that  $\Lambda_{\mathcal{F}}$  contains both more and less data than the linking system  $\mathcal{L}$ : Less in that only the left and right multiplications by  $S$  are defined (so that  $\Lambda_{\mathcal{F}}$  does not even have a partial group structure), but more in that the minimal  $\mathcal{F}$ -characteristic biset sees *all* the subgroups of  $S$  and not just the  $\mathcal{F}$ -centric ones. This suggests the possibility of using minimal characteristic bisets to avoid some of the nonfunctoriality of linking systems in future work.

Theorem C is a uniqueness statement about centric linking systems associated to  $\mathcal{F}$ . In Section 8 we establish a corresponding existence statement: Without reference to a centric linking system for  $\mathcal{F}$ , we set out to identify the  $\mathcal{F}$ -centric part of  $\Lambda_{\mathcal{F}}$ .

It turns out that the answer has a pleasingly simple form. If  $\varphi_i: P_i \xrightarrow{\cong} Q_i$ ,  $i = 1, 2$ , are  $\mathcal{F}$ -isomorphisms, we say that  $\varphi_1$  is equivalent to  $\varphi_2$  if there exist  $a, b \in S$  such that  $\varphi_2 = c_b \circ \varphi_1 \circ c_a$ .

**Theorem D** (Theorem 8.6). *The  $\mathcal{F}$ -centric part of  $\Lambda_{\mathcal{F}}$  has one  $(S, S)$ -orbit for each equivalence class of nonextendable isomorphisms of  $\mathcal{F}$ . The orbit corresponding to the class of  $\varphi: P \rightarrow Q$  is  $S \times_{(P, \varphi)} S$ .*

In [GRY], a general framework for computing the orbits of  $\Lambda_{\mathcal{F}}$  is developed as a special case of a much more general combinatorial argument. The advantage of the current Theorem D lies in the relative simplicity of the solution, along with the comparatively straightforward method used in the proof.

In Section 9, we close by considering the local group-theoretic properties of  $\Lambda_{\mathcal{F}}$ . Returning to the connection between point-stabilizers and conjugation from Axiom (0) of  $\mathcal{F}$ -characteristic bisets, we define notions of centralizer and normalizer subbisets. Given a subgroup  $P \in S$ , the  $\Lambda_{\mathcal{F}}$ -centralizer of  $P$  is the set of points  $C_{\Lambda_{\mathcal{F}}}(P) \subseteq \Lambda_{\mathcal{F}}$  satisfying  $a \cdot \omega = \omega \cdot a$  for all  $a \in P$ ; a similar definition made for the normalizer  $N_{\Lambda_{\mathcal{F}}}(P)$ . For  $P \leq S$  we also have notions of centralizer and normalizer fusion subsystems, denoted  $C_{\mathcal{F}}(P)$  and  $N_{\mathcal{F}}(P)$ , which are saturated fusion systems if  $P$  is fully  $\mathcal{F}$ -normalized (i.e.,  $|N_S(P)| \geq |N_S(\varphi P)|$  for all  $\mathcal{F}$ -morphisms  $\varphi: P \rightarrow S$ ). We show

**Theorem E** (Theorem 9.15). *If  $P$  is fully  $\mathcal{F}$ -normalized and additionally  $C_S(P) \leq P$ , then  $C_{\Lambda_{\mathcal{F}}}(P) = \Lambda_{C_{\mathcal{F}}(P)}$  and  $N_{\Lambda_{\mathcal{F}}}(P) = \Lambda_{N_{\mathcal{F}}(P)}$ . In other words, the centralizer of  $P$  in the minimal  $\mathcal{F}$ -characteristic biset is the minimal  $C_{\mathcal{F}}(P)$ -characteristic biset, and similarly for normalizers.*

In fact, we prove a more general statement in terms of Puig's notion of  $K$ -normalizers.

We interpret these results as saying that the minimal  $\mathcal{F}$ -characteristic biset is playing the role of a grouplike object inducing  $\mathcal{F}$  by conjugation, and that we are able to perform many group-theoretic operations in terms of  $\Lambda_{\mathcal{F}}$ .

We close with an open conjecture that the condition  $C_S(P) \leq P$  is not necessary in Theorem E. In other words, we conjecture that  $\mathcal{F}$ -centricity is not an essential concept in the world of minimal characteristic bisets, which would allow us avoid one of the most troublesome technical details in the study of  $p$ -local finite groups.

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## 2. FUSION SYSTEMS

The next few pages contain a very short introduction to fusion systems, which were originally introduced by Puig under the name “full Frobenius categories,” cf. [Pui1]. The aim is to introduce the terminology from the theory of fusion systems that will be used in the paper, and to establish the relevant notation. For a proper introduction to fusion systems see, for instance, Part I of “Fusion Systems in Algebra and Topology” by Aschbacher, Kessar and Oliver, [AKO].

**Definition 2.1.** A *fusion system*  $\mathcal{F}$  on a  $p$ -group  $S$ , is a category where the objects are the subgroups of  $S$ , and for all  $P, Q \leq S$  the morphisms must satisfy:

- (i) Every morphism  $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$  is an injective group homomorphism, and the composition of morphisms in  $\mathcal{F}$  is just composition of group homomorphisms.
- (ii)  $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q)$ , where

$$\text{Hom}_S(P, Q) = \{c_s \mid s \in N_S(P, Q)\}$$

is the set of group homomorphisms  $P \rightarrow Q$  induced by  $S$ -conjugation.

- (iii) For every morphism  $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$ , the group isomorphisms  $\varphi: P \rightarrow \varphi P$  and  $\varphi^{-1}: \varphi P \rightarrow P$  are elements of  $\text{Mor}_{\mathcal{F}}(P, \varphi P)$  and  $\text{Mor}_{\mathcal{F}}(\varphi P, P)$  respectively.

We also write  $\text{Hom}_{\mathcal{F}}(P, Q)$  or just  $\mathcal{F}(P, Q)$  for the morphism set  $\text{Mor}_{\mathcal{F}}(P, Q)$ ; and the group  $\mathcal{F}(P, P)$  of automorphisms is denoted by  $\text{Aut}_{\mathcal{F}}(P)$ .

The canonical example of a fusion system comes from a finite group  $G$  with a given  $p$ -subgroup  $S$ . The fusion system of  $G$  on  $S$ , denoted  $\mathcal{F}_S(G)$ , is the fusion system on  $S$  where the morphisms from  $P \leq S$  to  $Q \leq S$  are the homomorphisms induced by  $G$ -conjugation:

$$\text{Hom}_{\mathcal{F}_S(G)}(P, Q) := \text{Hom}_G(P, Q) = \{c_g \mid g \in N_G(P, Q)\},$$

A particular case is the fusion system  $\mathcal{F}_S(S)$  consisting only of the homomorphisms induced by  $S$ -conjugation.

Let  $\mathcal{F}$  be an abstract fusion system on  $S$ . We say that two subgroups  $P, Q \leq S$  are  $\mathcal{F}$ -conjugate, written  $P \sim_{\mathcal{F}} Q$ , if they are isomorphic in  $\mathcal{F}$ , i.e., there exists a group isomorphism  $\varphi \in \mathcal{F}(P, Q)$ .  $\mathcal{F}$ -conjugation is an equivalence relation, and the set of  $\mathcal{F}$ -conjugates to  $P$  is denoted by  $(P)_{\mathcal{F}}$ . The set of all  $\mathcal{F}$ -conjugacy classes of subgroups in  $S$  is denoted by  $Cl(\mathcal{F})$ . Similarly, we write  $P \sim_S Q$  if  $P$  and  $Q$  are  $S$ -conjugate, the  $S$ -conjugacy class of  $P$  is written  $(P)_S$  or just  $[P]$ , and we write  $Cl(S)$  for the set of  $S$ -conjugacy classes of subgroups in  $S$ . Since all  $S$ -conjugation maps are in  $\mathcal{F}$ , any  $\mathcal{F}$ -conjugacy class  $(P)_{\mathcal{F}}$  can be partitioned into disjoint  $S$ -conjugacy classes of subgroups  $Q \in (P)_{\mathcal{F}}$ .

We say that  $Q$  is  $\mathcal{F}$ - or  $S$ -subconjugate to  $P$  if  $Q$  is respectively  $\mathcal{F}$ - or  $S$ -conjugate to a subgroup of  $P$ . In the case where  $\mathcal{F} = \mathcal{F}_S(G)$ , then  $Q$  is  $\mathcal{F}$ -subconjugate to  $P$  if and only if  $Q$  is  $G$ -conjugate to a subgroup of  $P$ ; in this case the  $\mathcal{F}$ -conjugates of  $P$  are just those  $G$ -conjugates of  $P$  that are contained in  $S$ .

A subgroup  $P \leq S$  is said to be *fully  $\mathcal{F}$ -normalized* if  $|N_S P| \geq |N_S Q|$  for all  $Q \in (P)_{\mathcal{F}}$ ; similarly  $P$  is *fully  $\mathcal{F}$ -centralized* if  $|C_S P| \geq |C_S Q|$  for all  $Q \in (P)_{\mathcal{F}}$ .

**Definition 2.2.** A fusion system  $\mathcal{F}$  on  $S$  is said to be *saturated* if the following properties are satisfied for all  $P \leq S$ :

- (i) If  $P$  is fully  $\mathcal{F}$ -normalized, then  $P$  is fully  $\mathcal{F}$ -centralized, and  $\text{Aut}_S(P)$  is a Sylow  $p$ -subgroup of  $\text{Aut}_{\mathcal{F}}(P)$ .
- (ii) Every homomorphism  $\varphi \in \mathcal{F}(P, S)$  with  $\varphi(P)$  fully  $\mathcal{F}$ -centralized extends to a homomorphism  $\varphi \in \mathcal{F}(N_{\varphi} P, S)$ , where

$$N_{\varphi} := \{x \in N_S(P) \mid \exists y \in S: \varphi \circ c_x = c_y \circ \varphi\}$$

is the *extender* of  $\varphi$ .

The saturation axioms are a way of emulating the Sylow theorems for finite groups; in particular, whenever  $S$  is a Sylow  $p$ -subgroup of  $G$ , then the Sylow theorems imply that the induced fusion system  $\mathcal{F}_S(G)$  is saturated (see e.g. [AKO, Theorem 2.3]).

A particularly important consequence of the saturation axioms, which forms the basis for the key technical Lemma 4.3, is as follows:

**Lemma 2.3.** *Let  $\mathcal{F}$  be saturated. If  $P \leq S$  is fully normalized, then for each  $Q \in [P]_{\mathcal{F}}$  there exists a homomorphism  $\varphi \in \mathcal{F}(N_S Q, N_S P)$  with  $\varphi(Q) = P$ .*

For the proof, see Lemma 4.5 of [RS2] or Lemma 2.6(c) of [AKO].

### 3. BACKGROUND ON BISETS

In this section we recall the basic results about bisets – finite sets equipped with both a left and a right group action. In addition, we establish the necessary notation relating to bisets.

**Definition 3.1.** Let  $G$  and  $H$  be finite groups. A (free)  $(G, H)$ -biset  $\Omega$  is a set endowed with a free left  $H$ -action and a free right  $G$ -action, which commute:

$$h \cdot (\omega \cdot g) = (h \cdot \omega) \cdot g$$

When it is not clear from context which groups act on  $\Omega$ , we write  ${}_H\Omega_G$ .

Equivalently,  $\Omega$  is a left  $(H \times G)$ -set such that the restrictions of the action to  $H \times 1$  and  $1 \times G$  are free. This equivalence is formed by setting

$$(h, g) \cdot \omega = h \cdot \omega \cdot g^{-1}.$$

Given a  $(G, H)$ -biset  $\Omega$  the *opposite biset* is the  $(H, G)$ -biset  $\Omega^\circ$  with the same underlying set and with action defined by

$$g \cdot \omega^\circ \cdot h := h^{-1} \cdot \omega \cdot g^{-1}.$$

If  $G = H$  and  $\Omega \cong \Omega^\circ$  as  $(G, G)$ -bisets, we say  $\Omega$  is *symmetric*.

Denote by  $A_+(G, H)$  the monoid of isomorphism classes of  $(G, H)$ -bisets with disjoint union as addition. If  $\Omega \in A_+(G, H)$  and  $\Lambda \in A_+(H, K)$ , we define the  $(G, K)$ -biset  $\Lambda \circ \Omega$  to be  $\Lambda \times_H \Omega$ . With  $\circ$  as composition, the monoids  $A_+(G, H)$  form the morphism sets of a category where the objects are all finite groups. This is also the reason why a  $(G, H)$ -biset has  $G$  acting from the right and not the left, so that the composition order of bisets  $\Lambda \circ \Omega$  fits with the general convention for maps and morphisms.

The *point-stabilizer* of an element  $\omega$  in a  $(G, H)$ -biset  $\Omega$  is  $\text{Stab}_{H \times G}(\omega) \leq H \times G$ , the subgroup consisting of all pairs  $(h, g)$  such that  $h \cdot \omega = \omega \cdot g$ , or equivalently  $h \cdot \omega \cdot g^{-1} = \omega$ . A (*injective*)  $(G, H)$ -pair is a pair  $(K, \varphi)$  with  $K \leq G$  and  $\varphi: K \rightarrow H$  an injective group map. If  $(K, \varphi)$  is a  $(G, H)$ -pair, denote by  $[K, \varphi]$  the  $(G, H)$ -biset  $H \times_{(K, \varphi)} G := H \times G / (h, kg) \sim (h\varphi(k), g)$ . If we also denote by  $(K, \varphi)$  the *graph* of  $\varphi: K \rightarrow H$ :

$$(K, \varphi) := \{(\varphi(k), k) \in H \times G\},$$

then  $[K, \varphi] \cong (H \times G) / (K, \varphi)$  as  $H \times G$ -sets.

We will also refer to the graph  $(K, \varphi)$  as a *twisted diagonal (subgroup)*. In the case that  $G = H = S$  is a finite  $p$ -group,  $K = P \leq S$ , and  $\varphi \in \mathcal{F}(P, S)$  for a given fusion system  $\mathcal{F}$  on  $S$ , we will refer to  $(P, \varphi)$  as an  $\mathcal{F}$ -*twisted diagonal (subgroup)*.

The  $(G, H)$ -pairs  $(K, \varphi)$  and  $(L, \psi)$  are  $(G, H)$ -conjugate if there are elements  $g \in N_G(K, L)$  and  $h \in N_H(\varphi(K), \psi(L))$  such that  $L = {}^gK$  and

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & H \\ c_g \downarrow & & \downarrow c_h \\ L & \xrightarrow{\psi} & H \end{array}$$

commutes. This happens if and only if the twisted diagonals  $(K, \varphi)$  and  $(L, \psi)$  are conjugate as subgroups of  $H \times G$ .

**Fact 3.2.** *The  $(G, H)$ -bisets  $[K, \varphi]$  and  $[L, \psi]$  are isomorphic if and only if  $(K, \varphi)$  is  $(G, H)$ -conjugate to  $(L, \psi)$ . Moreover, every transitive  $(G, H)$ -biset is isomorphic to  $[K, \varphi]$  for some  $(G, H)$ -pair  $(K, \varphi)$ . In other words, if  $\Omega$  is a transitive  $(G, H)$ -biset, the stabilizer in  $H \times G$  of any point  $\omega \in \Omega$  is a subgroup of the form  $(K, \varphi)$ .*

Let  $S$  be a finite  $p$ -group and  $\mathcal{F}$  a saturated fusion system on  $S$ .

**Definition 3.3.** An  $(S, S)$ -biset  $\Omega$  is  $\mathcal{F}$ -generated if all point-stabilizers are  $\mathcal{F}$ -twisted diagonal subgroups.

$\Omega$  is  $\mathcal{F}$ -stable if for every  $(S, S)$ -pair  $(P, \varphi)$  and  $\mathcal{F}$ -isomorphisms  $\eta_1: Q \xrightarrow{\cong} P$  and  $\eta_2: \varphi P \xrightarrow{\cong} R$ , we have  $|\Omega^{(P, \varphi)}| = |\Omega^{(Q, \eta_2 \varphi \eta_1)}|$ .

**Definition 3.4.** An  $\mathcal{F}$ -semicharacteristic biset is an  $(S, S)$ -biset  $\Omega$  that satisfies:

- (i)  $\Omega$  is  $\mathcal{F}$ -generated.
- (ii)  $\Omega$  is  $\mathcal{F}$ -stable. When  $\Omega$  is  $\mathcal{F}$ -generated, it suffices to check that for each  $P \leq S$  and  $\varphi \in \mathcal{F}(P, S)$ , we have  $|\Omega^{(P, \varphi)}| = |\Omega^{(P, \iota_P^S)}| = |\Omega^{(\varphi(P), \varphi^{-1})}|$ , for  $\iota_P^S: P \rightarrow S$  the natural inclusion map.

$\Omega$  is an  $\mathcal{F}$ -characteristic biset if in addition

- (iii)  $|\Omega|/|S| \not\equiv 0 \pmod{p}$ .

*Example 3.5.* Suppose that  $S \in \text{Syl}_p(G)$  is equipped with the associated saturated fusion system  $\mathcal{F} := \mathcal{F}_S(G)$ . With left and right multiplication  $G$  is the  $(S, S)$ -biset  ${}_S G_S$ , which is always  $\mathcal{F}$ -characteristic:

For each  $g \in G$ ,  $\text{Stab}_{S \times S}(g) = (S \cap S^g, c_g)$ , hence  ${}_S G_S$  is  $\mathcal{F}$ -generated. If  $c_h \in \mathcal{F}(P, S)$  us any morphism in  $\mathcal{F}$ , the  $\mathcal{F}$ -twisted diagonal  $(P, c_h)$  is conjugate in  $G \times G$  to  $(P, \iota_P^S)$  and  $({}^h P, c_h^{-1})$ , so  $|\Omega^{(P, c_h)}| = |\Omega^{(P, \iota_P^S)}| = |\Omega^{({}^h P, c_h^{-1})}|$  and  $\Omega$  is  $\mathcal{F}$ -stable. Finally,  $S \in \text{Syl}_p(G)$  implies that  $p \nmid |{}_S G_S|/|S|$ .

**3.1. Some fixed point calculations.** In the rest of this section we aim to investigate fixed point sets of the form  $[Q, \psi]^{(P, \varphi)}$  that arise in our  $\mathcal{F}$ -characteristic bisets. This will in turn depend on the structure of the transporters  $N_{S \times S}((P, \varphi), (Q, \psi))$  via the formula

$$|[Q, \psi]^{(P, \varphi)}| = \frac{|N_{S \times S}((P, \varphi), (Q, \psi))|}{|(Q, \psi)|} = \frac{|N_{S \times S}((P, \varphi), (Q, \psi))|}{|Q|}.$$

To begin, suppose that  $(y, x) \in N_{S \times S}((P, \varphi), (Q, \psi))$ , so that for each  $p \in P$ , we have

$$(y, x)(\varphi(p), p)(y^{-1}, x^{-1}) = (\psi(q), q)$$

for some  $q \in Q$ . In particular, if  $xpx^{-1} = q$ , we have  $y\varphi(p)y^{-1} = \psi(q) = \psi(xpx^{-1})$ , so

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \varphi A \\ c_x \downarrow & & \downarrow c_y \\ B & \xrightarrow[\psi]{} & \psi B \end{array}$$

is a commuting diagram of group homomorphisms with  $x \in N_S(A, B)$  and  $y \in N_S(\varphi A, \psi B)$ . In particular,  $\psi \circ c_x \circ \varphi^{-1} = c_y: \varphi A \rightarrow \psi B$ , so that  $\psi \circ c_x \circ \varphi^{-1} \in \text{Hom}_S(\varphi A, \psi B)$ .

Conversely, consider an element  $x \in N_S(A, B)$  with  $\eta := \psi \circ c_x \circ \varphi^{-1} \in \text{Hom}_S(\varphi A, \psi B)$ . Then for every element  $y \in S$  such that  $c_y|_{\varphi A} = \eta$ , it is easy to see that we have a pair  $(y, x) \in N_{S \times S}((P, \varphi), (Q, \psi))$ , and that there are  $|C_S(\varphi A)|$  such  $y$  if there are any.

**Definition 3.6.** For  $A \xrightarrow{\varphi} \varphi A$  and  $B \xrightarrow{\psi} \psi B$  two morphisms of  $\mathcal{F}$ , set

$$N_{\varphi, \psi} := \{x \in N_S(A, B) \mid \psi \circ c_x \circ \varphi^{-1} \in \text{Hom}_S(\varphi A, \psi B)\}.$$

Note that the set  $N_{\varphi, \psi}$  is independent of the choice of the targets of  $\varphi$  and  $\psi$ , as is  $[[Q, \psi]^{(P, \varphi)}]$ . Since every morphism of  $\mathcal{F}$  factors uniquely as an isomorphism followed by an inclusion, we lose no data by focusing on just the isomorphisms of  $\mathcal{F}$ .

**Proposition 3.7.** Let  $P \xrightarrow{\varphi} \varphi P$  and  $Q \xrightarrow{\psi} \psi Q$  be two isomorphisms of  $\mathcal{F}$ .

- (a)  $N_{\varphi, \psi} = \text{pr}_2(N_{S \times S}((P, \varphi), (Q, \psi)))$  and  $N_{\varphi^{-1}, \psi^{-1}} = \text{pr}_1(N_{S \times S}((P, \varphi), (Q, \psi)))$  for  $\text{pr}_i$  the  $i$ th projection  $S \times S \rightarrow S$ ,  $i = 1, 2$ .
- (b)  $|N_{S \times S}((P, \varphi), (Q, \psi))| = |N_{\varphi, \psi}| \cdot |C_S(\varphi P)| = |N_{\varphi^{-1}, \psi^{-1}}| \cdot |C_S(P)|$ .
- (c)  $[[Q, \psi]^{(P, \varphi)}] = \frac{|N_{\varphi, \psi}| \cdot |C_S(\varphi P)|}{|Q|} = \frac{|N_{\varphi^{-1}, \psi^{-1}}| \cdot |C_S(P)|}{|Q|}$ .
- (d)  $N_{\varphi, \varphi} = N_{\varphi}$ , the standard extender of  $\varphi$ .
- (e)  $N_{\varphi, \psi}$  is naturally a free  $(N_{\varphi}, N_{\psi})$ -biset.

*Proof.* (a)-(d) are immediate from the preceding discussion. For (e), pick  $x \in N_{\varphi, \psi}$ ,  $n \in N_{\varphi}$ , and  $m \in N_{\psi}$ . We have

$$\begin{aligned} \psi \circ c_{mxn} \circ \varphi^{-1} &= (\psi \circ c_m \circ \psi^{-1}) \circ (\psi \circ c_x \circ \varphi^{-1}) \circ (\varphi \circ c_n \circ \varphi^{-1}) \\ &\in \text{Aut}_S(\psi B) \circ \text{Hom}_S(\varphi A, \psi B) \circ \text{Aut}_S(\varphi A) \\ &\subseteq \text{Hom}_S(\varphi A, \psi B), \end{aligned}$$

so  $m \cdot x \cdot n = mxn \in N_{\varphi, \psi}$ . Freeness is immediate.  $\square$

**Corollary 3.8.** Every  $\mathcal{F}$ -twisted diagonal subgroup  $(P, \varphi) \in S \times S$  is  $(\mathcal{F} \times \mathcal{F})$ -isomorphic to some  $(Q, \iota_Q^S)$  that is fully  $(\mathcal{F} \times \mathcal{F})$ -normalized. Moreover,  $(Q, \iota_Q^S)$  is fully  $(\mathcal{F} \times \mathcal{F})$ -normalized if and only if  $Q$  is fully  $\mathcal{F}$ -normalized.

*Proof.* That  $(P, \varphi)$  is  $(\mathcal{F} \times \mathcal{F})$ -conjugate with some  $(Q, \iota_Q^S)$  is clear from the definition of  $\mathcal{F} \times \mathcal{F}$ . Proposition 3.7 implies that  $|N_{S \times S}((P, \varphi))| = |N_{\varphi}| \cdot |C_S(\varphi P)|$ . It follows from the definition of the extender that  $|N_{\varphi}| \leq |N_S(P)|$  and  $N_{\iota_Q^S} = N_S(Q)$ . Therefore  $|N_{S \times S}((Q, \iota_Q^S))| = |N_S(Q)| \cdot |C_S(Q)|$ , and this is maximal in the  $(\mathcal{F} \times \mathcal{F})$ -class of  $(P, \varphi)$  precisely when  $Q$  is fully  $\mathcal{F}$ -normalized (as full  $\mathcal{F}$ -normalization implies full  $\mathcal{F}$ -centralization).  $\square$

Our first main goal is to parameterize the semicharacteristic bisets of  $\mathcal{F}$ . This will however require a short detour into the realm of sets with only one group action.

#### 4. THE FREE MONOID OF $\mathcal{F}$ -SETS

Let  $S$  be a finite  $p$ -group and  $\mathcal{F}$  a saturated fusion system on  $S$ . In analogy with the finite  $G$ -sets for a group  $G$ , this section studies a notion of  $\mathcal{F}$ -sets for a fusion system. We give a new proof of [Ree, Theorem A], that every finite  $\mathcal{F}$ -set decomposes uniquely, up to  $S$ -isomorphism, as a disjoint union of irreducible  $\mathcal{F}$ -sets. The key lemma is the same as in [Ree], but the main part of the proof is different: In the proof below, the decomposition is constructed explicitly by considering the actual  $\mathcal{F}$ -sets in play, while [Ree] relies on the structure of the Burnside ring of  $\mathcal{F}$  and linear algebra.



**Definition 4.1.** A finite  $\mathcal{F}$ -stable  $S$ -set, or just  $\mathcal{F}$ -set, is a finite set  $X$  with an action of  $S$  such that for all  $P \leq S$  and  $\varphi \in \mathcal{F}(P, S)$  the order of the fixed point sets of  $P$  and  $\varphi P$  are equal:  $|X^P| = |X^{\varphi P}|$ .

Let  $A_+(S)$  be the free commutative monoid of isomorphism classes of finite  $S$ -sets with disjoint union as addition, and let  $A_+(\mathcal{F}) \subseteq A_+(S)$  be the submonoid of isomorphism classes of  $\mathcal{F}$ -sets. Both  $A_+(S)$  and  $A_+(\mathcal{F})$  are semirings with Cartesian product as multiplication. Our goal in this section is to show that  $A_+(\mathcal{F})$  is a free commutative monoid.

**Definition 4.2.** The  $S$ -set  $X$  is  $\mathcal{F}$ -stable above level  $n$  if for any  $P \leq S$  with  $|P| \geq p^n$  and  $\varphi \in \mathcal{F}(P, S)$ , we have  $|X^P| = |X^{\varphi P}|$ . Clearly an  $S$ -set  $X$  is an  $\mathcal{F}$ -set if and only if  $X$  is  $\mathcal{F}$ -stable above level 0.

The following is the main technical result that implies the freeness of  $A_+(\mathcal{F})$ . We do not repeat the proof, but we do recall how it gives rise to an additive basis in the following.

**Lemma 4.3** ([Ree], Lemma 4.7). *Suppose that  $X$  is an  $S$ -set that is  $\mathcal{F}$ -stable above level  $n + 1$  and that the order of every stabilizer of every element of  $X$  is at least  $p^{n+1}$ . If  $P, Q \leq S$  are  $\mathcal{F}$ -conjugate subgroups of order  $p^n$  and  $Q$  is fully normalized in  $\mathcal{F}$ , then  $|X^Q| \geq |X^P|$ .*

**Notation 4.4.** Denote by  $Cl(S)$  the set of  $S$ -conjugacy classes of subgroups of  $S$ , and by  $Cl(\mathcal{F})$  the set of  $\mathcal{F}$ -conjugacy classes of subgroups. A class in  $Cl(S)$  will be denoted  $(P)_S$ , and a class in  $Cl(\mathcal{F})$  will be  $(P)_{\mathcal{F}}$ . Also, for  $(P)_S \in Cl(S)$ , let  $[P]$  denote the isomorphism class of the  $S$ -set  $S/P$ .

We now construct a collection of  $\mathcal{F}$ -sets satisfying particular structural properties. We will later show, in Corollary 4.7, that such  $\mathcal{F}$ -sets are irreducible and form a basis for  $A_+(\mathcal{F})$ .

**Theorem 4.5.** *For each  $P \leq S$  fully normalized in  $\mathcal{F}$ , there is an  $\mathcal{F}$ -set*

$$X_P = \coprod_{(Q)_S \in Cl(S)} c_Q \cdot [Q],$$

for  $c_Q \in \mathbb{Z}_{\geq 0}$ , that is uniquely determined as an  $S$ -set by requiring

- (i)  $c_P = 1$ ,
- (ii) If  $Q$  is fully normalized and  $c_Q \neq 0$ , then  $Q \cong_{\mathcal{F}} P$ .

**Remark 4.6.** The particular sets that we construct in the proof have additional properties:

- (iii) If  $c_Q \neq 0$ ,  $Q$  is  $\mathcal{F}$ -subconjugate to  $P$ .
- (iv) If  $P \cong_{\mathcal{F}} Q$  are both fully normalized, then  $X_P = X_Q$ , which contains exactly one copy of each orbit  $[P]$  and  $[Q]$ .

In Corollary 4.8, we argue that  $X_P$  in Theorem 4.5 is actually uniquely determined by properties (i) and (ii). Therefore  $X_P$  must have the structure specified in the proof below and satisfies (iii) and (iv).

Finally, we should note that while only (i)-(iv) will be used in this paper, much more can be said about the coefficients  $c_Q$  and the  $Q$ -fixed-point orders of  $X_P$ . The computations involved relate the combinatorics of the poset of subgroups of  $S$  to the shape of the category  $\mathcal{F}$  (i.e., which subgroups are made conjugate in the fusion system) together with  $p$ -local data concerning the orders of normalizers of certain subgroups. See [GRY] for more details.

*Proof.* We will begin with the  $S$ -set  $[P]$  and construct, in a minimal way, an  $\mathcal{F}$ -set containing  $[P]$ . We proceed level by level using Lemma 4.3 until we have a set which is  $\mathcal{F}$ -stable above level 0 and hence an  $\mathcal{F}$ -set.

Suppose that  $|P| = p^n$ . If  $Q \cong_{\mathcal{F}} P$  but  $Q \not\cong_S P$ ,  $[P]$  will not be  $\mathcal{F}$ -stable above level  $n$ :  $|[P]^P| = |N_S(P)|/|P|$  but  $|[P]^Q| = 0$ . To correct this while respecting (iii), we must add some number of copies of  $[Q]$ . Since  $|[Q]^Q| = |N_S(Q)|/|Q|$  and  $|Q| = |P|$ , it is easy to see that we must add  $\frac{|N_S(P)|}{|N_S(Q)|}$  copies of  $[Q]$  so that the number of  $Q$ -fixed points of the resulting  $S$ -set equals the number of  $P$ -fixed points. It follows easily that, if  $P = Q_1, Q_2, \dots, Q_a$  are representatives of the  $S$ -conjugacy classes of the  $\mathcal{F}$ -conjugacy class  $(P)_{\mathcal{F}}$ , the  $S$ -set

$$X_P^{(n)} := \prod_{i=1}^a \frac{|N_S(P)|}{|N_S(Q)|} \cdot [Q]$$

is an  $S$ -set,  $\mathcal{F}$ -stable above level  $n$ , that satisfies (i)-(iii). Note that had we used another fully normalized subgroup  $Q \cong_{\mathcal{F}} P$  instead of  $P$ , we would arrive at the same set:  $X_Q^{(n)} = X_P^{(n)}$ . Because the construction only depends on  $X_P^{(n)}$ ,  $X_Q = X_P$  and (iv) follows.

The trick then is to show that  $X_P^{(n)}$  is contained in an  $S$ -set  $X_P^{(n-1)}$  that satisfies (i)-(iii) and is  $\mathcal{F}$ -stable above level  $n-1$ ; the rest follows by obvious induction. So, suppose that  $Q \leq S$  is a subgroup of order  $p^{n-1}$ , and let  $R \in (Q)_{\mathcal{F}}$  be a fully normalized representative from the  $\mathcal{F}$ -conjugacy class. Lemma 4.3 implies that

$$\left| (X_P^{(n)})^Q \right| \leq \left| (X_P^{(n)})^R \right|.$$

The claim is that if the inequality is proper, we can add a certain number of copies of  $[Q]$  to  $X_P^{(n)}$  to force equality. Let  $\varphi \in \mathcal{F}(N_S(Q), N_S(R))$  be such that  $\varphi(Q) = R$ ; this exists by the saturation of  $\mathcal{F}$  and the assumption that  $R$  is fully  $\mathcal{F}$ -normalized.  $W_S(Q) := N_S(Q)/Q$  naturally acts on  $(X_P^{(n)})^Q$ . Similarly  $W_S(R)$  naturally acts on  $(X_P^{(n)})^R$ , and  $\varphi$  induces a map  $W_S(Q) \rightarrow W_S(R)$  and thus an action of  $W_S(Q)$  on  $(X_P^{(n)})^R$ .

Decompose

$$(X_P^{(n)})^R = (X_P^{(n)})_f^R \amalg (X_P^{(n)})_{nf}^R,$$

where  $(X_P^{(n)})_f^R$  is the subset of elements on which  $W_S(Q)$  acts freely and  $(X_P^{(n)})_{nf}^R$  are those elements on which  $W_S(Q)$  does not act freely. In other words,  $\omega \in (X_P^{(n)})_{nf}^R$  iff  $\omega \in (X_P^{(n)})^R$  and  $\text{Stab}_{W_S(Q)}(\omega) \neq 1$ . Similarly, decompose

$$(X_P^{(n)})^Q = (X_P^{(n)})_f^Q \amalg (X_P^{(n)})_{nf}^Q.$$

If  $\omega \in (X_P^{(n)})_{nf}^Q$ , let  $\overline{A} \leq W_S(Q)$  be the (nontrivial) stabilizer of  $\omega$  in  $W_S(Q)$ , and  $A \leq N_S(Q)$  the preimage of  $\overline{A}$ . Clearly  $A \leq \text{Stab}_S(\omega)$ , and  $|A| \geq p^n$ . In other words, every element of  $(X_P^{(n)})_{nf}^Q$  lies in  $(X_P^{(n)})^A$  for some  $A$  of order strictly greater than that of  $Q$ ; the same statement holds for  $(X_P^{(n)})_{nf}^R$ . By the inductive hypothesis,  $|(X_P^{(n)})^A| = |(X_P^{(n)})^{\varphi(A)}|$  for all such  $A$ , so we conclude

$$\left| (X_P^{(n)})_{nf}^Q \right| = \left| (X_P^{(n)})_{nf}^R \right|$$

by the same inclusion-exclusion argument in the proof of Lemma 4.3. Thus  $|(X_P^{(n)})^R| - |(X_P^{(n)})^Q| = |(X_P^{(n)})_f^R| - |(X_P^{(n)})_f^Q|$ , so in particular

$$c_Q := \frac{|(X_P^{(n)})^R| - |(X_P^{(n)})^Q|}{|W_S(Q)|} \in \mathbb{Z}_{\geq 0}.$$

This can be done for all subgroups  $Q \leq S$  of order  $p^{n-1}$ , with chosen representatives for each  $\mathcal{F}$ -conjugacy class.

From here it is easy to see that if we set

$$X_P^{(n-1)} = X_P^{(n)} \amalg \coprod_{\substack{(Q)_S \in Cl(S), \\ \text{s.t. } |Q|=p^{n-1}}} c_Q \cdot [Q],$$

then  $X_P^{(n-1)}$  satisfies (i)-(iii) and is  $\mathcal{F}$ -stable above level  $n-1$ , so we're done.  $\square$

**Corollary 4.7.** *Choose a fully normalized representative  $P^* \in (P)_{\mathcal{F}}$  from each class in  $Cl(\mathcal{F})$ . The  $\mathcal{F}$ -sets  $\{X_{P^*} \mid (P^*)_{\mathcal{F}} \in Cl(\mathcal{F})\}$  then form a basis for  $A_+(\mathcal{F})$ .*

*Proof.* Conditions (i) and (ii) imply that there can be no non-trivial  $\mathbb{Z}_{\geq 0}$ -linear (indeed,  $\mathbb{Z}$ -linear) relations amongst the  $X_{P^*}$ , so it suffices to show that every  $\mathcal{F}$ -set can be written as a sum of these.

Let  $X$  be an arbitrary  $\mathcal{F}$ -set, and pick a decomposition

$$X = \coprod_{(P)_S \in Cl(S)} c_P \cdot [P].$$

Consider the chosen representative  $P^* \in (P)_{\mathcal{F}}$  for each  $P \leq S$ , and set

$$Y := \coprod_{P^*} c_{P^*} \cdot [X_{P^*}].$$

Consider  $X - Y \in A(S)$ , in the Grothendieck group of  $A_+(S)$ ; if this can be shown to be 0,  $X$  will lie in  $\text{Span}_{\mathbb{Z}_{\geq 0}}\{X_{P^*} \mid (P^*)_{\mathcal{F}}\}$ , and we're done. We can extend  $|X^Q|$  linearly to the formal differences in  $A(S)$  in order to count generalized fixed points. If  $X - Y \neq 0$ , there is some subgroup  $Q \leq S$  of maximal order such that  $c_Q(X - Y) \neq 0$ . But for  $Q^*$  the chosen fully  $\mathcal{F}$ -normalized representative of  $(Q)_{\mathcal{F}}$ , we have  $c_{Q^*}(X - Y) = 0$  by construction, so

$$|(X - Y)^Q| = c_Q(X - Y) \cdot |W_S(Q)| \neq 0, \quad \text{while} \quad |(X - Y)^{Q^*}| = c_{Q^*}(X - Y) \cdot |W_S(Q)| = 0.$$

Hence  $|X^{Q^*}| = |Y^{Q^*}| = |Y^Q| \neq |X^Q|$  contradicting  $\mathcal{F}$ -stability of  $X$ .  $\square$

**Corollary 4.8.** *Suppose  $P \leq S$  is fully normalized. The  $\mathcal{F}$ -set  $X_P$  is uniquely determined by properties (i) and (ii), and is the unique minimal  $\mathcal{F}$ -set containing  $[P]$  as an orbit.*

*By Remark 4.6, it then follows that  $X_P$  depends only on the class  $(P)_{\mathcal{F}}$ , and for each fully normalized  $Q \in (P)_{\mathcal{F}}$  the  $\mathcal{F}$ -set  $X_P$  contains the orbit  $[Q]$  exactly once.*

*Proof.*  $X_P$  is part of a basis for  $A_+(\mathcal{F})$  as in Corollary 4.7. By properties (i) and (ii)  $X_P$  is the only basis element that contains  $[P]$  as an orbit, so every  $\mathcal{F}$ -set containing  $[P]$  has to contain a copy of the basis element  $X_P$ . It follows that  $X_P$  is the unique smallest  $\mathcal{F}$ -set containing  $[P]$ .  $\square$

This ends our detour to sets with only one group action, and we return to the world of bisets, in particular the  $\mathcal{F}$ -semicharacteristic ones.

5. THE PARAMETERIZATION OF SEMICARACTERISTIC BISETS OF  $\mathcal{F}$ 

In this section Theorem 5.3 parameterizes all the semicharacteristic bisets of  $\mathcal{F}$ . The method of approach is to apply the structure results of section 4 to the product fusion system  $\mathcal{F} \times \mathcal{F}$  and the monoid of  $(\mathcal{F} \times \mathcal{F})$ -sets.

**Lemma 5.1.** *Let  $(P, \varphi)$  and  $(Q, \psi)$  be two twisted diagonal subgroups of  $S \times S$ . Then  $(P, \varphi) \cong_{\mathcal{F} \times \mathcal{F}} (Q, \psi)$  if and only if there exist  $\mathcal{F}$ -isomorphisms  $\eta_1 \in \mathcal{F}(P, Q)$  and  $\eta_2 \in \mathcal{F}(\varphi P, \psi Q)$  such that*

$$\begin{array}{ccc} P & \xrightarrow[\cong]{\eta_1} & Q \\ \varphi \downarrow & & \downarrow \psi \\ \varphi P & \xrightarrow[\cong]{\eta_2} & \psi Q \end{array}$$

*commutes. In particular, any twisted diagonal subgroup  $(P, \varphi) \leq S \times S$  with  $\varphi \in \mathcal{F}(P, S)$  is  $(\mathcal{F} \times \mathcal{F})$ -isomorphic to every  $(Q, \iota_Q^S)$  where  $Q \cong_{\mathcal{F}} P$ .*

*Proof.* Obvious from the definition of  $\mathcal{F} \times \mathcal{F}$ .  $\square$

**Proposition 5.2.** *A (free)  $(S, S)$ -biset  $\Omega$  is  $\mathcal{F}$ -stable if and only if  $\Omega$  is  $(\mathcal{F} \times \mathcal{F})$ -stable when viewed as an  $(S \times S)$ -set.*

*Proof.* A morphism of  $\mathcal{F} \times \mathcal{F}$  is the restriction of a morphism  $(\varphi, \psi)$ , for  $\varphi \in \mathcal{F}(P, S)$  and  $\psi \in \mathcal{F}(Q, S)$ , to some subgroup of  $P \times Q$ . As  $\Omega$  is bifree, the only subgroups of  $S \times S$  with nonempty fixed point sets are twisted diagonals  $(P, \varphi)$ . By Lemma 5.1  $(P, \varphi) \cong_{\mathcal{F} \times \mathcal{F}} (Q, \psi)$  iff there exist  $\mathcal{F}$ -isomorphisms  $\eta_1: Q \xrightarrow{\cong} P$  and  $\eta_2: \varphi P \xrightarrow{\cong} \psi Q$  such that  $\psi = \eta_2 \varphi \eta_1$ . Hence the  $(\mathcal{F} \times \mathcal{F})$ -stability condition is equivalent to the condition for  $\mathcal{F}$ -stable bisets.  $\square$

**Theorem 5.3.** *Let  $\mathcal{F}$  be a saturated fusion system on  $S$ . For each  $\mathcal{F}$ -conjugacy class of subgroups  $(P)_{\mathcal{F}} \in Cl(\mathcal{F})$  there is an associated  $\mathcal{F}$ -semicharacteristic biset  $\Omega_P$ : Supposing  $P$  is fully normalized,  $\Omega_P$  is the smallest  $\mathcal{F}$ -semicharacteristic biset containing  $[P, \iota_P^S]$ . The sets  $\Omega_P$ , taken together, form an additive basis for the free monoid of semicharacteristic bisets of  $\mathcal{F}$ . Moreover, an  $\mathcal{F}$ -semicharacteristic biset*

$$\Omega = \coprod_{(P)_{\mathcal{F}} \in Cl(\mathcal{F})} c_P \cdot \Omega_P$$

*is  $\mathcal{F}$ -characteristic if and only if  $p \nmid c_S$ .*

*Proof.* Pick a representative  $P \in (P)_{\mathcal{F}}$  such that  $(P, \iota_P^S)$  is fully normalized in  $\mathcal{F} \times \mathcal{F}$ ; we can choose such a  $P$  by Corollary 3.8. Define  $\Omega_P$  to be the unique  $(\mathcal{F} \times \mathcal{F})$ -set corresponding to the subgroup  $(P, \iota_P^S) \leq S \times S$  defined in Theorem 4.5, and by Corollary 4.8 this is the smallest  $(\mathcal{F} \times \mathcal{F})$ -set containing  $[P, \iota_P^S]$ . Property (iii) of remark 4.6 states that every point-stabilizer of  $\Omega_P$  is  $(\mathcal{F} \times \mathcal{F})$ -subconjugate to the diagonal  $(P, \iota_P^S)$ , so  $\Omega_P$  is  $\mathcal{F}$ -generated and hence semicharacteristic for  $\mathcal{F}$ .

The collection  $\{\Omega_P\}_{(P)_{\mathcal{F}} \in Cl(\mathcal{F})}$  forms a basis for a submonoid of  $A_+(\mathcal{F} \times \mathcal{F})$ , as it is part of the basis for the entire monoid  $A_+(\mathcal{F} \times \mathcal{F})$ . The submonoid spanned by the  $\Omega_P$  consists only of those  $(\mathcal{F} \times \mathcal{F})$ -sets whose point-stabilizers are  $\mathcal{F}$ -twisted diagonal subgroups. By the same downward induction in the proof of Corollary 4.7, we see that every  $(\mathcal{F} \times \mathcal{F})$ -set with point-stabilizers  $\mathcal{F}$ -twisted diagonal subgroups lies in this submonoid. Finally, being  $\mathcal{F}$ -semicharacteristic is equivalent to having  $\mathcal{F}$ -twisted diagonal point-stabilizers and being  $(\mathcal{F} \times \mathcal{F})$ -stable (Proposition 5.2), thus proving that the  $\Omega_P$  form a basis for the monoid of semicharacteristic bisets of  $\mathcal{F}$ .

To prove the last claim, it is enough to show that  $p$  divides  $|\Omega_P/S| = |\Omega_P|/|S|$  if and only if  $P \neq S$ . As  $|[P, \varphi]| = |S \times S|/|P|$ , it is clear that  $p$  divides  $|[P, \varphi]|/|S|$  if and only if  $|P| < |S|$ . As every point-stabilizer of  $\Omega_P$  is  $(\mathcal{F} \times \mathcal{F})$ -subconjugate to  $[P, \iota_P^S]$ , it follows that  $|\Omega_P|$  is divisible by  $|[P, \iota_P^S]|/|S|$  which is divisible by  $p$  if  $P \neq S$ . Therefore the choice of the number  $c_P$  has no effect on whether or not  $\Omega$  is  $\mathcal{F}$ -characteristic when  $P \neq S$ .

Finally,  $\Omega_S$  can be decomposed

$$\Omega_S = \left( \coprod_{[\alpha] \in \text{Out}_{\mathcal{F}}(S)} [S, \alpha] \right) \amalg \left( \coprod_{\substack{|P| < |S| \\ \varphi \in \mathcal{F}(P, S)}} c_{P, \varphi} [P, \varphi] \right)$$

for constants  $c_{P, \varphi} \in \mathbb{Z}_{\geq 0}$ . Each term  $[S, \alpha]$  has  $|S|$  elements, while  $p|S| \mid |[P, \varphi]|$  when  $|P| < |S|$ . Therefore  $|\Omega_S|/|S| \equiv |\text{Out}_{\mathcal{F}}(S)| \not\equiv 0$  modulo  $p$  by the saturation axioms of fusion systems.  $\square$

**Corollary 5.4.** *Each fusion system has a unique minimal  $\mathcal{F}$ -characteristic biset  $\Lambda = \Lambda_{\mathcal{F}}$ , in the sense that if  $\Omega$  is any  $\mathcal{F}$ -characteristic biset for  $\mathcal{F}$ , up to isomorphism we have  $\Lambda \subseteq \Omega$ .*

*Proof.* Define  $\Lambda_{\mathcal{F}} = \Omega_S$  in the notation of Theorem 5.3; the rest is immediate.  $\square$

**Proposition 5.5.** *Each of the  $\mathcal{F}$ -semicharacteristic basis elements  $\Omega_P$  is a symmetric  $(S, S)$ -biset. Hence every  $\mathcal{F}$ -semicharacteristic biset is symmetric.*

*Proof.*  $\Omega_P^o$  is  $\mathcal{F}$ -semicharacteristic and contains the orbit  $[P, \iota_P^S]^o \cong [P, \iota_P^S]$ . Because  $\Omega_P$  is the smallest  $\mathcal{F}$ -semicharacteristic biset containing  $[P, \iota_P^S]$ , we must have  $\Omega_P \subseteq \Omega_P^o$ . Size considerations, or applying  $(-)^o$  again, tell us that equality  $\Omega_P = \Omega_P^o$  holds.  $\square$

## 6. MINIMAL CHARACTERISTIC BISETS OF CONSTRAINED FUSION SYSTEMS

We know that any finite group  $G$  is a  $\mathcal{F}$ -characteristic biset for its associated fusion system  $\mathcal{F}_S(G)$ ; see example 3.5. For a constrained fusion system  $\mathcal{F}$ , a saturated fusion system that contains a normal and  $\mathcal{F}$ -centric subgroup, Broto-Castellana-Grodal-Levi-Oliver have shown that  $\mathcal{F}$  has a unique minimal group *model*. This section shows that the model for a contained fusion system is not just a  $\mathcal{F}$ -characteristic biset, it is always isomorphic to the minimal  $\mathcal{F}$ -characteristic biset for the fusion system.

**Proposition 6.1.** *Let  $G$  be a finite group with  $\mathcal{F} = \mathcal{F}_S(G)$  and  $N \leq S$  a normal subgroup of  $G$ . If  $(P, \varphi)$  is a point-stabilizer of the  $(S, S)$ -biset  ${}_S G_S$ , then  $N \leq P$ .*

*Proof.* Pick  $g \in G$  and suppose that  $(Q, \psi)$  stabilizes  $g$ , so that  $g \cdot q = \psi(q) \cdot g$  for all  $q \in Q$ . Therefore  $\psi(q) = gqg^{-1}$  and  $g \in N_G(Q, S)$ . As  $N \trianglelefteq G$ , we have  $g \in N_G(N \cdot Q, N \cdot S)$ . As  $N \leq S$ , if we set  $P = N \cdot Q$  we have that conjugation by  $g$  induces a map  $\varphi \in \mathcal{F}(P, S)$ . Thus  $gpg^{-1} = \varphi(p)$  for all  $p \in P$ , or  $g \cdot p = \varphi(p) \cdot g$ . Thus  $g \in ({}_S G_S)^{(P, \varphi)}$  and  $(P, \varphi) \leq \text{Stab}_{S \times S}(g)$ . The result follows.  $\square$

Note that in Proposition 6.1, we do not assume that  $S \in \text{Syl}_p(G)$ , only that  $S$  contains a normal  $p$ -subgroup of  $G$ . If we additionally require that  $S$  is Sylow in  $G$ , there is a canonical choice for  $N \trianglelefteq G$ , namely the largest normal  $p$ -subgroup of  $G$ .

**Notation 6.2.** If  $G$  is a finite group,  $O_p(G)$  denotes the largest normal  $p$ -subgroup of  $G$ , and  $O_{p'}(G)$  the largest normal  $p'$ -subgroup.

**Corollary 6.3.** *Let  $G$  be a finite group with  $S \in \text{Syl}_p(G)$  and  $\mathcal{F} = \mathcal{F}_S(G)$ . If the  $\mathcal{F}$ -characteristic biset  ${}_S G_S$  decomposes as  $\coprod_{(P) \in \mathcal{F} \in \text{Cl}(\mathcal{F})} c_P \cdot \Omega_P$ , then  $c_P \neq 0$  implies  $O_p(G) \leq P$ .*

*Proof.* By Proposition 6.1 and the fact that  $O_p(G) = \bigcap_{S' \in \text{Syl}_p(G)} S'$ , we see that every point-stabilizer of  ${}_S G_S$  is of the form  $(P, \varphi)$  with  $O_p(G) \leq P$ . As the  $\mathcal{F}$ -semicharacteristic biset  $\Omega_Q$  contains the  $(S, S)$ -biset  $[Q, \iota_Q^S]$ , which has an element with stabilizer  $(Q, \iota_Q^S)$ , it follows that  $c_Q = 0$  for all  $Q \not\geq O_p(G)$ . The result follows.  $\square$

There is a general version of Proposition 6.1 and Corollary 6.3 for abstract fusion systems (Proposition 9.11), but the proof is more involved.

**Definition 6.4.** Let  $G$  be a finite group.

- $G$  is  $p'$ -reduced if  $O_{p'}(G) = 1$ .
- If  $G$  is  $p'$ -reduced,  $G$  is  $p$ -constrained if  $C_G(O_p(G)) \leq O_p(G)$ .

Note that  $G/O_{p'}(G)$  is always  $p'$ -reduced, so that we might define a general  $G$  to be  $p$ -constrained if  $G/O_{p'}(G)$  is  $p$ -constrained. We will not make use of this definition here.

**Definition 6.5.** Let  $\mathcal{F}$  be a saturated fusion system on  $S$ . We write  $O_p(\mathcal{F})$  for the largest normal subgroup of  $\mathcal{F}$ . Thus,  $O_p(\mathcal{F}) \trianglelefteq S$  is maximal subject to the requirement that for every  $\varphi \in \mathcal{F}(P, Q)$ , there is some extension  $\tilde{\varphi} \in \mathcal{F}(P \cdot O_p(\mathcal{F}), Q \cdot O_p(\mathcal{F}))$  such that  $\tilde{\varphi}(O_p(\mathcal{F})) = O_p(\mathcal{F})$ .

A saturated fusion system  $\mathcal{F}$  is *constrained* if  $O_p(\mathcal{F})$  is  $\mathcal{F}$ -centric, or equivalently if  $C_S(O_p(\mathcal{F})) \leq O_p(\mathcal{F})$ . A *model* for the constrained fusion system  $\mathcal{F}$  is a finite group  $M$  that is  $p'$ -reduced,  $p$ -constrained, contains  $S$  as a Sylow  $p$ -subgroup, and  $\mathcal{F} = \mathcal{F}_S(M)$ .

**Theorem 6.6** ([BCG<sup>+</sup>, Proposition C]). *Every constrained fusion system has a unique model.*

We then reach the main result of this section describing the model of a constrained fusion system as a  $\mathcal{F}$ -characteristic biset.

**Theorem 6.7.** *Let  $\mathcal{F}$  be a constrained fusion system on  $S$  and  $M$  the model for  $\mathcal{F}$ . Then the  $(S, S)$ -biset  ${}_S M_S$  is the unique minimal  $\mathcal{F}$ -characteristic biset  $\Lambda_{\mathcal{F}}$  of  $\mathcal{F}$ .*

*Proof.* We will show (1) if  $(P, \iota_P^S)$  is a point-stabilizer of  ${}_S M_S$ , then  $P = S$ , and (2) any two elements of  ${}_S M_S$  whose stabilizers are  $(S, \text{id})$  lie in the same  $(S, S)$ -orbit. In light of the characterization of basis element of  $A_+(\mathcal{F} \times \mathcal{F})$  from Theorem 4.5, the result will follow immediately from these facts and Theorem 5.3: (1) shows that  ${}_S M_S$  is a multiple of  $\Omega_S = \Lambda_{\mathcal{F}}$ , and (2) shows that  ${}_S M_S$  contains at most one copy of  $\Omega_S$ .

(1): Pick  $m \in {}_S M_S$ ,  $\text{Stab}_{S \times S}(m) = (P, \iota_P^S)$ . By Proposition 6.1, we may assume that  $O_p(G) \leq P$ . Thus for any  $m \in ({}_S M_S)^{(P, \iota_P^S)}$ , we have  $m \cdot a = a \cdot m$  for all  $a \in P$ . Therefore  $m \in C_M(P) \leq C_M(O_p(G)) \leq O_p(G) \leq P$ , so that  $m \in S$  and  $m$  induces the automorphism  $c_m \in \text{Inn}(S)$ . Thus for all  $s \in S$ ,  $m \cdot s = c_m(s) \cdot m$ , or  $m \in ({}_S M_S)^{(S, c_m)}$ . As  $(P, \iota_P^S)$  was already identified as the stabilizer of  $m$ , we conclude  $P = S$  and  $m \in Z(S)$ .

(2): Suppose that  $m, m' \in {}_S M_S$  are two elements with point-stabilizer  $(S, \text{id})$ . By the last conclusion of part (1), we have  $m, m' \in Z(S) \leq S$ , and as  ${}_S S_S$  is a transitive subbiset of  ${}_S M_S$ , the result follows.  $\square$

## 7. CENTRIC MINIMAL CHARACTERISTIC BISETS ARISING FROM LINKING SYSTEMS

In this section we describe the relationship between a centric linking system  $\mathcal{L}$  for a saturated fusion system and the minimal  $\mathcal{F}$ -characteristic biset.

For  $\mathcal{F}$ -centric subgroups  $P, Q \leq S$ , identify  $N_S(P, Q)$  with its image in  $\mathcal{L}(P, Q)$ . The composite of  $\mathfrak{g} \in \mathcal{L}(P, Q)$  and  $\mathfrak{h} \in \mathcal{L}(Q, R)$  will be written  $\mathfrak{h} \cdot \mathfrak{g} \in \mathcal{L}(P, R)$ .

We recall the extension result for morphisms of linking systems:

**Theorem 7.1** ([OV]). *Pick  $\mathfrak{g} \in \mathcal{L}_{\text{iso}}(P, Q)$  and normal supergroups  $P \trianglelefteq \tilde{P}$ ,  $Q \trianglelefteq \tilde{Q}$ . If for every  $\tilde{p} \in \tilde{P}$  we have  $\mathfrak{g} \cdot \tilde{p} \cdot \mathfrak{g}^{-1} \in \tilde{Q}$ , then  $\mathfrak{g}$  has a unique extension  $\tilde{\mathfrak{g}} \in \mathcal{L}(\tilde{P}, \tilde{Q})$ .*

**Corollary 7.2.** *Let  $\mathfrak{g} \in \mathcal{L}_{\text{iso}}(P, Q)$  be an isomorphism of  $\mathcal{L}$ . The following are equivalent:*

- (a)  $\mathfrak{g}$  is nonextendable.
- (b)  $(\mathfrak{g}^{-1} \cdot N_S(Q) \cdot \mathfrak{g}) \cap N_S(P) = P$ .
- (c)  $(\mathfrak{g} \cdot N_S(P) \cdot \mathfrak{g}^{-1}) \cap N_S(Q) = Q$ .

*Proof.* (a) $\Leftrightarrow$ (b):  $\mathfrak{g}$  can always extend to  $(\mathfrak{g}^{-1} \cdot N_S(Q) \cdot \mathfrak{g}) \cap N_S(P)$  by Theorem 7.1. On the other hand, if  $\mathfrak{g}$  is extendable, without loss of generality we may assume that  $\mathfrak{g}$  extends to some  $\tilde{\mathfrak{g}} \in \mathcal{L}_{\text{iso}}(\tilde{P}, \tilde{Q})$  with  $P \trianglelefteq \tilde{P}$ . Then for any  $\tilde{p} \in \tilde{P}$ , the diagram

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{\mathfrak{g}}} & \tilde{Q} \\ \tilde{p} \downarrow & & \downarrow c_{\tilde{\mathfrak{g}}}(\tilde{p}) \\ \tilde{P} & \xrightarrow{\tilde{\mathfrak{g}}} & \tilde{Q} \end{array}$$

commutes in  $\mathcal{L}$ . Here  $c_{\tilde{\mathfrak{g}}} \in \mathcal{F}(\tilde{P}, \tilde{Q})$  is the image of  $\tilde{\mathfrak{g}}$  in the underlying fusion system. On restriction, this diagram becomes

$$\begin{array}{ccc} P & \xrightarrow{\mathfrak{g}} & Q \\ \tilde{p} \downarrow & & \downarrow c_{\tilde{\mathfrak{g}}}(\tilde{p}) \\ P & \xrightarrow{\mathfrak{g}} & Q \end{array}.$$

Thus  $\mathfrak{g}^{-1} \cdot c_{\tilde{\mathfrak{g}}}(\tilde{p}) \cdot \mathfrak{g} = \tilde{p} \in (\mathfrak{g}^{-1} \cdot N_S(Q) \cdot \mathfrak{g}) \cap N_S(P)$ , and the result follows.

(a) $\Leftrightarrow$ (c): If  $\tilde{\mathfrak{g}}$  is an extension of  $\mathfrak{g}$ , then  $\tilde{\mathfrak{g}}^{-1}$  is an extension of  $\mathfrak{g}^{-1}$ . Thus the equivalence of (a) and (c) is the same as that of (a) and (b), with  $\mathfrak{g}^{-1}$  in the role of  $\mathfrak{g}$ .  $\square$

One can use this result to prove that the equivalence relation on the set of isomorphisms of  $\mathcal{L}$  generated by restriction has a particularly nice structure.

**Theorem 7.3** ([Che], Lemma A.8). *Let  $\mathfrak{g}_1 \in \mathcal{L}_{\text{iso}}(P_1, Q_1)$  and  $\mathfrak{g}_2 \in \mathcal{L}_{\text{iso}}(P_2, Q_2)$  be two isomorphisms that can be connected by a chain of extensions and restrictions. Then there is an isomorphism  $\mathfrak{h}$  with source containing  $\langle P_1, P_2 \rangle$  and target containing  $\langle Q_1, Q_2 \rangle$  such that the restriction of  $\mathfrak{h}$  to  $P_i$  is  $\mathfrak{g}_i$ ,  $i = 1, 2$ .*

*In particular, each equivalence class of isomorphisms of  $\mathcal{L}$  contains a unique maximal element  $\mathfrak{k}$ , in the sense that every element of that class is a restriction of  $\mathfrak{k}$ . This unique maximal element is of necessity nonextendable, and each nonextendable isomorphism appears as the maximal element of a different class.*

**Notation 7.4.** Let  $\mathfrak{I}$  denote the set of nonextendable isomorphisms of  $\mathcal{L}$ . By Theorem 7.3 every morphism of  $\mathcal{L}$  is then the restriction of a unique isomorphism in  $\mathfrak{I}$ .

( $\mathfrak{I}$  is in fact the underlying set of Chermak's partial group version of a linking system.)

**Lemma 7.5.** *The set  $\mathfrak{I}$  carries a natural  $(S, S)$ -biset structure.*

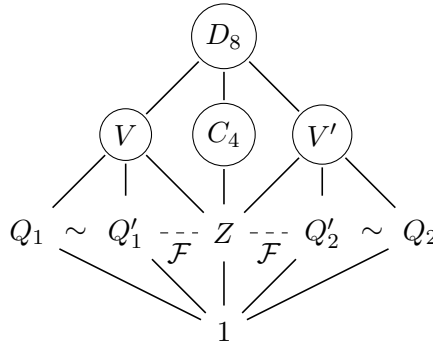
*Proof.* Pick  $P \xrightarrow{g} Q \in \mathfrak{I}$  and  $a, b \in S$ . Define  $a \cdot g \cdot b \in \mathcal{L}(b^{-1}P, {}^aQ)$  to be the composite

$$b^{-1}P \xrightarrow{b} P \xrightarrow{g} Q \xrightarrow{a} {}^aQ.$$

Pick some  $n \in N_S({}^aQ)$  such that  $(agb)^{-1} \cdot n \cdot (agb) \in N_S(b^{-1}P)$ , then  $g^{-1}(a^{-1}na)g \in N_S(P)$ . As  $g \in \mathfrak{I}$  is nonextendable, Corollary 7.2 forces  $a^{-1}na \in Q$ , so  $n \in {}^aQ$  and  $a \cdot g \cdot b$  is nonextendable.  $\square$

It is not the case that  ${}_S\mathfrak{I}_S$  is an  $\mathcal{F}$ -characteristic set, as the example of  $\mathcal{F}_{D_8}(A_6)$  demonstrates. The main failing is that the elements of  $\mathfrak{I}$ , being morphisms in  $\mathcal{L}$ , only see the  $\mathcal{F}$ -centric subgroups.

*Example 7.6.* Inside  $\mathcal{F} := \mathcal{F}_{D_8}(A_6)$ , the Sylow 2-subgroup  $D_8$  has the following subgroup diagram:



Each sign  $\sim$  in the diagram indicates that the two subgroups are conjugate in  $D_8$ , and each  $\text{---}\mathcal{F}\text{---}$  indicates that the subgroups are conjugate in  $\mathcal{F}$  but not in  $D_8$ . Finally, the circles indicate the  $\mathcal{F}$ -centric subgroups of  $D_8$ .

The fusion system  $\mathcal{F}$  is generated by an outer automorphism  $\alpha: V \rightarrow V$  sending  $Q_1$  to  $Z$  and an outer automorphism  $\beta: V' \rightarrow V'$  sending  $Q_2$  to  $Z$ . Let  $\mathcal{L}$  be the centric linking system for  $\mathcal{F}$ . The  $\mathcal{L}$ -automorphisms of  $D_8$  are the elements of  $D_8$  itself, and these form a single  $(D_8, D_8)$ -orbit of type  $[D_8, id]$ . All  $\mathcal{L}$ -automorphisms of  $C_4$  extend to  $D_8$ , hence they do not contribute to the biset  ${}_S\mathfrak{I}_S$ . Of the 24  $\mathcal{L}$ -automorphisms of  $V$  only 8 of them extend to  $D_8$ ; the remaining 16 form a single  $(D_8, D_8)$ -orbit of type  $[V, \alpha]$ . Similarly the nonextendable  $\mathcal{L}$ -automorphisms of  $V'$  produce a biset orbit  $[V', \beta]$ .

The entire biset  ${}_S\mathfrak{I}_S$  of nonextendable  $\mathcal{L}$ -isomorphisms is thus isomorphic to

$${}_S\mathfrak{I}_S \cong [D_8, id] + [V, \alpha] + [V', \beta].$$

This however is not all of the characteristic biset for  $\mathcal{F}$ .  $\Lambda_{\mathcal{F}}$  receives two additional orbits from the non- $\mathcal{F}$ -centric subgroups:

$$\Lambda_{\mathcal{F}} = [D_8, id] + [V, \alpha] + [V', \beta] + [Q_1, \beta^{-1}\alpha] + [Q_2, \alpha^{-1}\beta].$$

Note that that  $\beta^{-1}\alpha: Q_1 \rightarrow Q_2$  is nonextendable, as is its inverse  $\alpha^{-1}\beta: Q_2 \rightarrow Q_1$ , so each must be represented as a point-stabilizer in  $\Lambda_{\mathcal{F}}$ .

**Definition 7.7.** An  $\mathcal{F}$ -centric semicharacteristic biset is an  $\mathcal{F}$ -generated  $(S, S)$ -biset  $\Omega$  with all point-stabilizers of the form  $(P, \varphi)$  with  $P$  an  $\mathcal{F}$ -centric subgroup, and such that for all  $\mathcal{F}$ -centric subgroups  $P$  and  $\varphi \in \mathcal{F}(P, S)$ ,  $|\Omega^{(P, \varphi)}| = |\Omega^{(P, \iota_P^S)}| = |\Omega^{(\varphi P, \varphi^{-1})}|$ . If we also have  $|\Omega|/|S| \not\equiv 0 \pmod p$ , we say that  $\Omega$  is a  $\mathcal{F}$ -centric characteristic biset.



*Remark 7.8.* Each  $\mathcal{F}$ -centric semicharacteristic biset  $\Omega$  is by assumption  $\mathcal{F}$ -stable on all the  $\mathcal{F}$ -centric subgroups of  $S$ . By adding additional orbits  $[Q, \psi]$  with  $Q$  non-centric, as in the construction of Theorem 4.5, we can construct a  $\mathcal{F}$ -semicharacteristic biset from  $\Omega$ . Conversely, any semicharacteristic biset for  $\mathcal{F}$  can be truncated, by removing all orbits  $[Q, \psi]$  with  $Q$  non-centric, to give a  $\mathcal{F}$ -centric semicharacteristic biset.

This provides a 1-to-1 correspondence between the centric (semi)characteristic bisets for  $\mathcal{F}$  and those (semi)characteristic bisets of the form  $\sum_{\mathcal{F}\text{-centric } (P)\mathcal{F}} c_P \cdot \Omega_P$  with  $c_P \in \mathbb{Z}_{\geq 0}$ .

**Theorem 7.9.**  ${}_S\mathfrak{I}_S$  is an  $\mathcal{F}$ -centric characteristic biset. Moreover, it is the unique minimal  $\mathcal{F}$ -centric characteristic biset for  $\mathcal{F}$ , and thus is the  $\mathcal{F}$ -centric part of the minimal characteristic biset for  $\mathcal{F}$ .

*Proof.* Suppose that  $(R, \chi)$  is the stabilizer of  $P \xrightarrow{\mathfrak{g}} Q \in \mathfrak{I}$ , so that  $\chi(r) \cdot \mathfrak{g} \cdot r^{-1} = \mathfrak{g}$  for all  $r \in R$ . The definition of the  $(S, S)$ -action forces  $R \leq N_S(P)$  and  $\chi(R) \leq N_S(Q)$ .  $\mathfrak{g}$  is nonextendable, so Corollary 7.2 implies  $R \leq P$  and  $\chi = c_{\mathfrak{g}}|_R$ . As  $(P, c_{\mathfrak{g}})$  fixes  $\mathfrak{g}$ , it follows that  $(R, \chi) = (P, c_{\mathfrak{g}})$ , so every point-stabilizer of  ${}_S\mathfrak{I}_S$  is a  $\mathcal{F}$ -twisted diagonal subgroup whose source is  $\mathcal{F}$ -centric.

We now demonstrate  $\mathcal{F}$ -stability on the  $\mathcal{F}$ -centrics. Let  $P$  be an  $\mathcal{F}$ -centric subgroup and  $(P, \varphi)$  an  $\mathcal{F}$ -twisted diagonal subgroup; we claim  $|({}_S\mathfrak{I}_S)^{(P, \varphi)}| = |Z(P)|$ . If  $A \xrightarrow{\mathfrak{h}} B \in ({}_S\mathfrak{I}_S)^{(P, \varphi)}$ , then  $\varphi(p) \cdot \mathfrak{h} \cdot p^{-1} = \mathfrak{h}$  for all  $p \in P$ , so the above argument gives  $P \leq A$  and  $\varphi = c_{\mathfrak{h}}|_P$ . In other words, there is a natural bijection between the fixed points of  $(P, \varphi)$  and the elements of  $\mathfrak{I}$  that restrict to  $\varphi$ . As every morphism of  $\mathcal{L}$  is epi and mono, an element of  $\mathfrak{I}$  is uniquely determined by its restriction and conversely, so the number of  $(P, \varphi)$ -fixed points is the number of isomorphisms in  $\mathcal{L}$  with source  $P$  that project to  $\varphi$  in  $\mathcal{F}$ . By the linking system axioms there are  $|Z(P)|$  such isomorphisms, proving the claim.

Finally, we show that  ${}_S\mathfrak{I}_S$  is minimal. If  $\text{Stab}_{S \times S}(\mathfrak{g}) = (P, \iota_P^S)$ , we must have  $c_{\mathfrak{g}} = \text{id}_P$ , which is only nonextendable when  $P = S$ . Thus if  $(P, \iota_P^S)$  is a stabilizer, we must have  $P = S$ . Finally, as  $|[S, \text{id}]^{(S, \text{id})}| = |Z(S)| = |({}_S\mathfrak{I}_S)^{(S, \text{id})}|$ , we conclude that there is exactly one orbit with stabilizer  $(S, \text{id})$ , and we are done.  $\square$

## 8. THE LINKING-SYSTEM-FREE CENTRIC MINIMAL CHARACTERISTIC BISSET

In this section we determine the minimal  $\mathcal{F}$ -centric characteristic biset for a saturated fusion system  $\mathcal{F}$  in purely fusion-theoretic terms without assuming the existence of a linking system for  $\mathcal{F}$ . The key for the argument is Puig's result, here recorded as Proposition 8.3 and Corollary 8.5, describing the degree to which a morphism between  $\mathcal{F}$ -centric subgroups has unique extensions.

*Remark 8.1.* Fix  $\varphi \in \mathcal{F}_{\text{iso}}(P, Q)$ . For  $a, b \in S$ , set  $\psi := c_a \circ \varphi \circ c_b \in \mathcal{F}_{\text{iso}}({}^{b^{-1}}P, {}^aQ)$ . If  ${}^{b^{-1}}P \leq {}^{b^{-1}}\tilde{P}$  and  $\tilde{\psi} \in \mathcal{F}({}^{b^{-1}}\tilde{P}, S)$  extends  $\psi$ , then  $c_a^{-1} \circ \tilde{\psi} \circ c_b^{-1} \in \mathcal{F}(\tilde{P}, S)$  extends  $\varphi$ . Thus  $\varphi$  is nonextendable if and only if  $c_a \circ \varphi \circ c_b$  is nonextendable for all  $a, b \in S$ .

**Notation 8.2.** Let  $\mathcal{I}$  be a set of representatives of the equivalence classes of nonextendable  $\mathcal{F}$ -isomorphisms between  $\mathcal{F}$ -centric subgroups of  $S$ , where  $\varphi \sim \varphi'$  if there exist  $a, b \in S$  such that  $\varphi' = c_a \circ \varphi \circ c_b$ .

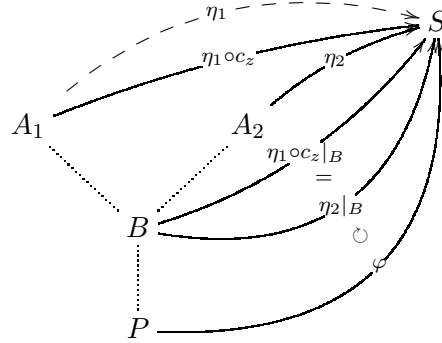
**Proposition 8.3.** [Pui1, Proposition 3.3] *Let  $P \leq Q \leq S$  be two  $\mathcal{F}$ -centric subgroups. If  $\psi_1, \psi_2: Q \rightarrow S$  are such that  $\psi_1|_P = \psi_2|_P$ , then there is some  $z \in Z(P)$  such that  $\psi_2 = \psi_1 \circ c_z|_Q \in \mathcal{F}(Q, S)$ .*

*Remark 8.4.* In fact, Puig's formulation deals with  $\mathcal{F}$ -quasicentric subgroups ("nilcentralized" in his terminology), a more general class of subgroups than the  $\mathcal{F}$ -centrics. The original statements is: If  $P \leq Q \leq S$  are  $\mathcal{F}$ -quasicentric subgroups with  $\psi_1, \psi_2 \in \mathcal{F}(Q, S)$  such that  $\psi_1|_P = \psi_2|_P =: \varphi \in \mathcal{F}(P, S)$  and  $\varphi P$  is fully  $\mathcal{F}$ -centralized, then there is some  $z \in C_S(\varphi P)$  such that  $\psi_2 = c_z \circ \psi_1 \in \mathcal{F}(Q, S)$ . In the case that  $P$  is  $\mathcal{F}$ -centric, we have  $C_S(\varphi P) = Z(\varphi P)$ . Thus  $z = \varphi(z') = \psi_1(z')$  for some  $z' \in Z(P)$ , and  $c_z \circ \psi_1 = \psi_1 \circ c_{z'}$ , and we recover the above formulation.

**Corollary 8.5.** *If  $P \leq S$  is  $\mathcal{F}$ -centric then each  $\varphi \in \mathcal{F}(P, S)$  has a unique nonextendable extension, up to precomposition with conjugation by elements of  $Z(P)$ . In other words, if  $\psi_1 \in \mathcal{F}(Q_1, S)$  and  $\psi_2 \in \mathcal{F}(Q_2, S)$  are both nonextendable extensions of  $\varphi$ , then  $Q_1 = Q_2$  and there is some  $z \in Z(P)$  such that  $\psi_2 = \psi_1 \circ c_z$ .*

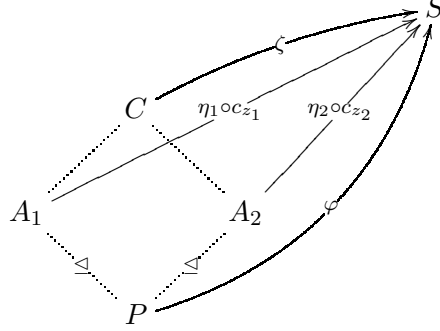
*Proof.* We break the proof into three steps.

(1) *Conjugate uniqueness on intersections:* First suppose that we have two (possibly extendable) extensions  $\eta_1 \in \mathcal{F}(A_1, S)$  and  $\eta_2 \in \mathcal{F}(A_2, S)$  of  $\varphi \in \mathcal{F}(P, S)$ , and set  $B := A_1 \cap A_2$ . Then  $\eta_1|_B$  and  $\eta_2|_B$  are two extensions of  $\varphi$  with the same source  $B$ , so by Proposition 8.3 there is some  $z \in Z(P)$  such that  $\eta_2|_B = \eta_1 \circ c_z|_B$ . Thus, up to precomposition with conjugation by a central element of  $P$ , we may assume that any two extensions of  $\varphi$  agree wherever both are defined.



(2) *Existence and conjugate uniqueness of normal extensions:* Suppose now we have two extensions (still possibly extendable)  $\eta_1 \in \mathcal{F}(A_1, S)$  and  $\eta_2 \in \mathcal{F}(A_2, S)$  of  $\varphi \in \mathcal{F}(P, S)$ , and that  $P \trianglelefteq A_i$ ,  $i = 1, 2$ . Set  $C := \langle A_1, A_2 \rangle \leq N_S(P)$ . Recall that  $N_\varphi$ , the extender of  $\varphi$ , is the largest subgroup of  $N_S(P)$  for which there exists an extension of  $\varphi$  (because all subgroups in sight are  $\mathcal{F}$ -centric). By assumption, we have  $A_i \leq N_\varphi$ ,  $i = 1, 2$ . Hence  $C \leq N_\varphi$  as well, and there is some  $\zeta \in \mathcal{F}(C, S)$  that extends  $\varphi$ . As  $\zeta|_{A_i}$  and  $\eta_i$  are two morphisms in  $\mathcal{F}(A_i, S)$  that extend  $\varphi$ , Proposition 8.3 implies that there is some  $z_i \in Z(P)$  such that  $\eta_i \circ c_{z_i} = \zeta|_{A_i} \in \mathcal{F}(A_i, S)$ . Thus, up to composition with conjugation by a central element of  $P$ , the extensions  $\eta_1$  and  $\eta_2$  of  $\varphi$  have a common extension, at least when  $P$  is

normal in the sources of the  $\eta_i$ .



(3) *General uniqueness*: Finally, suppose that  $\psi \in \mathcal{F}(Q, S)$  is a *nonextendable* extension of  $\varphi$ , and  $\chi \in \mathcal{F}(R, S)$  is some extension. We will show that  $R \leq Q$  and that there is some  $z \in Z(P)$  such that  $\psi|_R = \chi \circ c_z \in \mathcal{F}(R, S)$ . Clearly this will imply the overall result.

Set  $B := Q \cap R$ . By step (1), we may assume that  $\psi|_B = \chi|_B$ . If  $B = Q$ , the nonextendability of  $\psi$  forces  $R = Q$ , and we have our result.

Let us therefore induct on the index  $[Q : B]$ . If  $B \leq Q$ , then either  $R \leq Q$  (and we're done) or  $B$  is properly contained in both  $N_Q(B)$  and  $N_R(B)$ . Set  $C := \langle N_Q(B), N_R(B) \rangle$ ; by the second step, there is some  $\eta \in \mathcal{F}(C, S)$  that also extends  $\varphi$ , and such that  $\eta|_{N_R(B)} = \chi \circ c_z|_{N_R(B)}$  for some  $z \in Z(P)$ . As  $[Q : C \cap Q] \leq [Q : N_Q(B)] < [Q : B]$ , our inductive hypothesis gives us that  $C \leq Q$ . In particular,  $N_R(B) \leq Q$ . If  $B = R \cap Q$  is properly contained in  $R$  this yields a contradiction, so we conclude  $R \leq Q$ , and we're done.  $\square$

**Theorem 8.6.** *The  $(S, S)$ -biset  $\Omega = \coprod_{(\psi: Q \rightarrow Q') \in \mathcal{I}} [Q, \psi]$  is the minimal  $\mathcal{F}$ -centric characteristic biset.*

*Proof.* Clearly  $\Omega$  is  $\mathcal{F}$ -generated, all point-stabilizers are  $\mathcal{F}$ -twisted diagonals with source  $\mathcal{F}$ -centric subgroups,  $\Omega$  has precisely one orbit isomorphic to  $[S, \text{id}_S]$ , and no other orbits are isomorphic to  $[P, \iota_P^S]$ . Moreover, the only orbits of order  $|S|$  are those of the form  $[S, \alpha]$  for  $\alpha \in \text{Out}_{\mathcal{F}}(S)$ . Therefore  $|\Omega|/|S| \equiv |\text{Out}_{\mathcal{F}}(S)| \not\equiv 0$  modulo  $p$ . Thus the only thing to do is show that  $\Omega$  is  $\mathcal{F}$ -stable on  $\mathcal{F}$ -centric subgroups. If  $P \leq S$  is  $\mathcal{F}$ -centric,  $\varphi \in \mathcal{F}(P, S)$ , and  $\omega \in \Omega$  has point-stabilizer  $(Q, \psi)$ , it is clear that  $\omega \in \Omega^{(P, \varphi)}$  if and only if  $(P, \varphi) \leq (Q, \psi)$ , i.e.,  $P \leq Q$  and  $\psi$  is an extension of  $\varphi$ .

Proposition 8.5 implies that any two elements of  $\Omega^{(P, \varphi)}$  must lie in the same  $(S, S)$ -orbit of  $\Omega$ : If  $\omega_i \in \Omega^{(P, \varphi)}$  have stabilizers  $(Q_i, \psi_i)$ ,  $i = 1, 2$ , then  $Q_1 = Q_2$  and there is some  $z \in Z(P)$  such that  $\psi_2 = \psi_1 \circ c_z$ . As  $(Q_1, \psi_1)$  and  $(Q_1, \psi_1 \circ c_z)$  are  $(S \times S)$ -conjugate, and  $\Omega$  has no two orbits that are isomorphic, we conclude that  $\omega_1$  and  $\omega_2$  lie in the same orbit. Thus  $|\Omega^{(P, \varphi)}| = |[Q, \psi]^{(P, \varphi)}|$ , with  $\psi$  our chosen representative in  $\mathcal{I}$  of the unique nonextendable extension of  $\varphi$ . By Proposition 3.7 (c),

$$|[Q, \psi]^{(P, \varphi)}| = \frac{|N_{\varphi, \psi}| \cdot |C_S(\varphi P)|}{|Q|} = \frac{|N_{\varphi, \psi}|}{|Q|} \cdot |Z(P)|.$$

We claim that  $N_{\varphi, \psi} = Q$ , so that order of the fixed point set is  $|Z(P)|$ . As this order depends only on the source of  $\varphi$ , it will follow that  $\Omega$  is  $\mathcal{F}$ -stable on  $\mathcal{F}$ -centrics.

Recall that  $N_{\varphi, \psi} = \{x \in N_S(P, Q) \mid \psi \circ c_x \circ \varphi^{-1} \in \text{Hom}_S(\varphi P, \psi Q)\}$ . If  $q \in Q$ , we have

$$\psi \circ c_q \circ \varphi^{-1} = c_{\psi(q)} \circ \psi \circ \varphi^{-1} = c_{\psi(q)} \circ \varphi \circ \varphi^{-1} = c_{\psi(q)} \in \text{Hom}_S(\varphi P, \psi Q).$$

Therefore  $Q \subseteq N_{\varphi, \psi}$ .

For the other direction, fix  $x \in N_{\varphi, \psi}$ . There is some  $y \in N_S(\varphi P, \psi Q)$  such that

$$\psi \circ c_x \circ \varphi^{-1} = c_y \in \text{Hom}_S(\varphi P, \psi Q), \text{ hence } c_y^{-1} \circ \psi \circ c_x|_P = \varphi \in \mathcal{F}(P, \varphi P).$$

Thus  $c_y^{-1} \circ \psi \circ c_x \in \mathcal{F}(x^{-1}Q, S)$  and  $\psi \in \mathcal{F}(Q, S)$  are two extensions of  $\varphi$ ; by Proposition 8.5 we have  $x^{-1}Q \leq Q$ , hence  $x^{-1}Q = Q$ , and there is  $z \in Z(P)$  such that  $c_y^{-1} \circ \psi \circ c_x = \psi \circ c_z$ . Rewriting this as  $\psi \circ c_{zx^{-1}} \circ \psi^{-1} = c_{y^{-1}} \in \text{Aut}_S(\psi Q)$ . We have  $zx^{-1} \in N_S(Q)$ , so that  $zx^{-1} \in N_{\psi, \psi} = N_\psi$ . As  $\psi$  is nonextendable (with target an  $\mathcal{F}$ -centric, and hence fully  $\mathcal{F}$ -centralized, subgroup),  $N_\psi = Q$  by the extension axiom for saturated fusion systems. Thus  $zx^{-1} \in Q$ . As  $z \in Z(P) \leq P \leq Q$ , we conclude  $x \in Q$ , and the proof is complete.  $\square$

## 9. $K$ -NORMALIZERS

For any saturated fusion system  $\mathcal{F}$  on  $S$  and any fully  $\mathcal{F}$ -normalized subgroup  $P \leq S$  we can consider the associated normalizer fusion system  $N_{\mathcal{F}}(P)$ ; similarly for fully  $\mathcal{F}$ -centralized  $P$  and the centralizer fusion system  $C_{\mathcal{F}}(P)$ . We might wonder whether it is possible to construct a minimal characteristic biset for  $N_{\mathcal{F}}(P)$  if we are given a minimal characteristic biset for  $\mathcal{F}$ . In this section we introduce a normalizer subbiset  $N_{\Omega}(P) \subseteq \Omega$  for a subgroup  $P$  (resp., centralizer subbiset  $C_{\Omega}(P)$ ) and show that in many cases this will be a characteristic biset for  $N_{\mathcal{F}}(P)$  (resp.  $C_{\mathcal{F}}(P)$ ).

**Definition 9.1.** For  $P \leq S$  and a subgroup  $K \leq \text{Aut}(P)$ , we define the following concepts:

- The  $K$ -normalizer of  $P$  in  $S$  is the group  $N_S^K(P) = \{n \in N_S(P) \mid c_n|_P \in K\}$ .
- If  $Q \leq S$  is isomorphic to  $P$  via an abstract group isomorphism  $\varphi: P \rightarrow Q$ , set  ${}^{\varphi}K = \{\varphi \circ \alpha \circ \varphi^{-1} \mid \alpha \in K\} \leq \text{Aut}(Q)$ .
- $P$  is *fully  $K$ -normalized in  $\mathcal{F}$*  if for all  $\varphi \in \mathcal{F}(P, S)$  we have  $|N_S^K(P)| \geq |N_S^{{}^{\varphi}K}(\varphi P)|$ .
- The  $K$ -normalizer fusion system is the fusion system  $N_{\mathcal{F}}^K(P)$  on  $N_S^K(P)$  with morphisms given by

$$\begin{aligned} \text{Hom}_{N_{\mathcal{F}}^K(P)}(A, B) &= \{\varphi \in \mathcal{F}(A, B) \mid \exists \tilde{\varphi} \in \mathcal{F}(PA, PB) \\ &\quad \text{s.t. } \tilde{\varphi}|_A = \varphi, \tilde{\varphi}P = P, \text{ and } \tilde{\varphi}|_P \in K\}. \end{aligned}$$

**Proposition 9.2** ([Pui1, Propositions 2.12 & 2.15]). *Let  $P \leq S$  and  $K \leq \text{Aut}(P)$ . Then  $P$  is fully  $K$ -normalized in  $\mathcal{F}$  if and only if  $P$  is fully  $\mathcal{F}$ -centralized and  $\text{Aut}_S(P) \cap K \in \text{Syl}_p(\mathcal{F}(P) \cap K)$ .*

*Furthermore, if  $P$  is fully  $K$ -normalized in  $\mathcal{F}$ , then  $N_{\mathcal{F}}^K(P)$  is a saturated fusion system on  $N_S^K(P)$ .*

*Example 9.3.* We have the following special cases of  $K$ -normalizers:

- If  $K = \{\text{id}_P\}$  is the trivial subgroup of  $\text{Aut}(P)$ , then  $N_{\mathcal{F}}^K(P) = C_{\mathcal{F}}(P)$  is the *centralizer fusion subsystem* of  $P$ , whose underlying  $p$ -group is  $C_S(P)$ .
- If  $K = \text{Aut}(P)$  is the full automorphism group of  $P$ , then  $N_{\mathcal{F}}^K(P) = N_{\mathcal{F}}(P)$  is the *normalizer fusion subsystem* of  $P$ , whose underlying  $p$ -group is  $N_S(P)$ .

In the following, we let  $\Omega$  be some fixed  $\mathcal{F}$ -semicharacteristic  $(S, S)$ -biset.

**Definition 9.4.** For any  $P \leq S$  and  $K \leq \text{Aut}(P)$ , the  $K$ -normalizer of  $P$  in  $\Omega$  is

$$N_{\Omega}^K(P) := \{\omega \in \Omega \mid \text{Stab}_{S \times S}(\omega) = (Q, \psi), P \leq Q, \psi P = P, \text{ and } \psi|_P \in K\} \subseteq \Omega.$$

If  $K = \{\text{id}_P\}$ , we denote the resulting *centralizer* of  $P$  in  $\Omega$  by  $C_{\Omega}(P)$ ; if  $K = \text{Aut}(P)$ , the *normalizer* of  $P$  in  $\Omega$  will be written  $N_{\Omega}(P)$ .

*Remark 9.5.* If  $\omega \in \Omega$  has stabilizer  $(Q, \psi)$ , then for any  $a, b \in S$ , we have

$$\text{Stab}_{S \times S}(a \cdot \omega \cdot b) = (Q^b, c_a \circ \psi \circ c_b).$$

In particular,  $N_\Omega^K(P)$  need not be an  $(S, S)$ -biset.

**Lemma 9.6.**  $N_\Omega^K(P)$  is naturally an  $(N_S^K(P), N_S^K(P))$ -biset.

*Proof.*  $n, m \in N_S^K(P)$ ,  $\omega \in N_\Omega^K(P)$ . If  $\text{Stab}_{S \times S}(\omega) = (Q, \psi)$ , then  $\text{Stab}_{S \times S}(n \cdot \omega \cdot m) = (Q^m, c_n \circ \psi \circ c_m)$ . It is clear that  $P \leq Q^m$  and  $(c_n \circ \psi \circ c_m)(P) = P$ , and as each  $c_m|_P$ ,  $c_n|_P$ , and  $\psi|_P$  lie in  $K$  it follows that  $n \cdot \omega \cdot m \in N_\Omega^K(P)$ , and the claim is proved.  $\square$

**Notation 9.7.** For the rest of this section,  $P$  denotes some chosen subgroup of  $S$ , and we set  $N := N_S^K(P)$  and  $\mathcal{N} := N_{\mathcal{F}}^K(P)$ .

As the first step in deciding whether  $N_\Omega^K(P)$  is  $\mathcal{N}$ -characteristic, we describe the  $(N, N)$ -stabilizer of each element in  $N_\Omega^K(P)$ .

**Lemma 9.8.**  $\omega \in N_\Omega^K(P)$ . If  $\text{Stab}_{S \times S}(\omega) = (Q, \psi)$ , then  $\text{Stab}_{N \times N}(\omega) = (N \cap Q, \psi|_{N \cap Q})$ .

*Proof.* The only nontrivial part is that  $(N \cap Q, \psi|_{N \cap Q}) \leq N \times N$ , i.e., that  $\psi(N \cap Q) \leq N$ . If  $n \in N \cap Q$ , then  $n \in N_S(P)$ , so  $\psi(n) \in N_S(\psi P) = N_S(P)$ . In addition we have  $c_{\psi(n)}|_P = (\psi \circ c_n \circ \psi^{-1})|_P \in K$ , so  $\psi(n) \in N$  follows.  $\square$

**Lemma 9.9.**  $A, B, C \leq N$ ,  $\varphi \in \mathcal{N}_{\text{iso}}(A, B)$ , and  $\psi \in \mathcal{N}_{\text{iso}}(A, C)$ . The number of extensions of  $\varphi$  to  $\tilde{\varphi} \in \mathcal{N}_{\text{iso}}(PA, PB)$  equals the number of extensions of  $\psi$  in  $\mathcal{N}_{\text{iso}}(PA, PC)$ .

Dually, if  $A, B, C \leq N$ ,  $\varphi \in \mathcal{N}_{\text{iso}}(A, C)$ , and  $\psi \in \mathcal{N}_{\text{iso}}(B, C)$ , then the number of extensions of  $\varphi$  to  $\tilde{\varphi} \in \mathcal{N}_{\text{iso}}(PA, PC)$  equals the number of extensions of  $\psi$  in  $\mathcal{N}_{\text{iso}}(PB, PC)$ .

*Proof.* Any extension of  $\varphi \in \mathcal{N}_{\text{iso}}(A, B)$  with source  $PA$  (whose existence is guaranteed by the definition of  $\mathcal{N}$ ) has image  $PB$ . If  $\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathcal{N}_{\text{iso}}(PA, PB)$  are extensions of  $\varphi$ , then  $\tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1 \in \mathcal{N}(PA)$  and  $\tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1|_A = \varphi^{-1} \circ \varphi = \text{id}_A$ . Let  $G \leq \mathcal{N}(PA)$  be the group of  $\mathcal{N}$ -automorphisms of  $PA$  that restrict to the identity on  $A$ , so that  $G$  acts transitively on the set of lifts of  $\varphi$  by precomposition. This action is free, so the number of extensions of  $\varphi$  to an  $\mathcal{N}$ -isomorphism with source  $PA$  is  $|G|$ . The same is true for any other  $\mathcal{N}$ -isomorphism with source  $A$ , and the result is proved.

The dual statement is proved by replacing each isomorphism with its inverse.  $\square$

**Proposition 9.10.** If  $\Omega$  is an  $\mathcal{F}$ -semicharacteristic biset, then  $N_\Omega^K(P)$  is an  $\mathcal{N}$ -semicharacteristic  $(N, N)$ -biset.

*Proof.*  $N_\Omega^K(P)$  is  $\mathcal{N}$ -generated: This is immediate from the definition and Lemma 9.8.

$N_\Omega^K(P)$  is  $\mathcal{N}$ -stable: If  $A \leq N$  and  $\varphi \in \mathcal{N}(A, N)$ , we want to show that

$$\left| (N_\Omega^K(P))^{(A, \varphi)} \right| = \left| (N_\Omega^K(P))^{(A, \iota_A^N)} \right| = \left| (N_\Omega^K(P))^{(\varphi A, \varphi^{-1})} \right|.$$

If  $P \leq A$ , we claim that  $(N_\Omega^K(P))^{(A, \varphi)} = \Omega^{(A, \varphi)}$ . The containment  $\subseteq$  is obvious. Suppose that  $\omega \in \Omega^{(A, \varphi)}$ ,  $\text{Stab}_{S \times S}(\omega) = (Q, \psi)$ . We must have  $(A, \varphi) \leq (Q, \psi)$ , so that  $A \leq Q$  and  $\psi$  is an extension of  $\varphi$ . It follows that  $\psi P = \varphi P = P$  and  $\psi|_P = \varphi|_P \in K$ , so  $\omega \in N_\Omega^K(P)$ , as claimed.  $\Omega$  is  $\mathcal{F}$ -stable, so  $N_\Omega^K(P)$  is  $\mathcal{F}$ -stable on those twisted diagonal subgroups  $(A, \varphi)$  such that  $P \leq A$ .

In general, given  $A \leq N$  and an isomorphism  $\varphi \in \mathcal{N}_{\text{iso}}(A, B)$ , we consider the set  $\{\tilde{\varphi}_i \in \mathcal{N}_{\text{iso}}(PA, PB)\}_{i=1}^n$  of extensions of  $\varphi$  to an isomorphism with source  $PA$  (which must necessarily have target  $PB$ ). We then claim

$$(N_{\Omega}^K(P))^{(A, \varphi)} = \coprod_{i=1}^n \Omega^{(PA, \tilde{\varphi}_i)}.$$

The union is disjoint: If there are  $1 \leq i, j \leq n$  such that  $\omega \in \Omega^{(PA, \tilde{\varphi}_i)} \cap \Omega^{(PA, \tilde{\varphi}_j)}$ , then for all  $x \in PA$ ,  $\tilde{\varphi}_i(x) \cdot \omega \cdot x^{-1} = \omega = \tilde{\varphi}_j(x) \cdot \omega \cdot x^{-1}$ . The left  $S$ -action on  $\Omega$  is free, so  $\tilde{\varphi}_i(x) = \tilde{\varphi}_j(x)$  for all  $x \in PA$ , hence  $i = j$ .

The equality holds: For  $\omega \in \Omega^{(PA, \tilde{\varphi}_i)}$ , we have  $\omega \in N_{\Omega}^K(P)$  because  $\tilde{\varphi}_i P = P$  and  $\tilde{\varphi}_i|_P \in K$ .  $(A, \varphi) \leq (PA, \tilde{\varphi}_i)$ , implies  $\omega \in (N_{\Omega}^K(P))^{(A, \varphi)}$ . Conversely, if  $\omega \in (N_{\Omega}^K(P))^{(A, \varphi)}$ ,  $\text{Stab}_{S \times S}(\omega) = (Q, \psi) \leq S \times S$ , then by definition of  $N_{\Omega}^K(P)$  we have  $P \leq Q$ ,  $\psi P = P$ , and  $\psi|_P \in K$ . Thus  $\psi(PA) \leq N$  and  $\psi|_{PA} \in \mathcal{N}_{\text{iso}}(PA, PB)$  is an extension of  $\varphi$ . Therefore there is some  $\tilde{\varphi}_i$  such that  $\omega \in \Omega^{(PA, \tilde{\varphi}_i)}$ , proving the reverse containment.

Putting these claims together:

$$\begin{aligned} |(N_{\Omega}^K(P))^{(A, \varphi)}| &= \left| \coprod_{i=1}^n \Omega^{(PA, \tilde{\varphi}_i)} \right| = \sum_{i=1}^n |\Omega^{(PA, \tilde{\varphi}_i)}| = n \cdot |\Omega^{(PA, \iota_{PA}^S)}| \\ &= n \cdot |(N_{\Omega}^K(P))^{(PA, \iota_{PA}^N)}|. \end{aligned}$$

The third equality uses the  $\mathcal{F}$ -stability of  $\Omega$ ; the fourth our observation that  $N_{\Omega}^K(P)$  is  $\mathcal{N}$ -stable on those subgroups that contain  $P$ . Note in particular that we have described  $|(N_{\Omega}^K(P))^{(A, \varphi)}|$  as depending solely on the number of extensions  $n$  of  $\varphi$  to an isomorphism in  $\mathcal{N}$  with source  $PA$ . By Lemma 9.9, this number depends not on  $\varphi \in \mathcal{N}_{\text{iso}}(A, B)$ , but only on the source  $A$ . It follows that  $|(N_{\Omega}^K(P))^{(A, \varphi)}| = |(N_{\Omega}^K(P))^{(A, \iota_A^N)}|$ . The dual result of Lemma 9.9 implies that the number  $n$  also can be seen to depend only on the target of the isomorphism; as  $\text{id}_A$  and  $\varphi^{-1} \in \mathcal{N}_{\text{iso}}(\varphi A, A)$  have the same target, it follows that  $|(N_{\Omega}^K(P))^{(A, \iota_A^N)}| = |(N_{\Omega}^K(P))^{(\varphi A, \varphi^{-1})}|$ , and the  $\mathcal{N}$ -stability of  $N_{\Omega}^K(P)$  is proved.  $\square$

Aside: The method of the proof of Proposition 9.10 can be used to prove the following useful structure theorem for the minimal  $\mathcal{F}$ -characteristic biset  $\Lambda_{\mathcal{F}}$ :

**Proposition 9.11.** *If  $(Q, \psi)$  is a point-stabilizer of  $\Lambda_{\mathcal{F}}$ , then  $O_p(\mathcal{F}) \leq Q$ .*

*Proof.* Let  $\Xi$  be the  $(S, S)$ -biset obtained by applying the  $(\mathcal{F} \times \mathcal{F})$ -stabilization process of Theorem 4.5 to  $[S, \text{id}_S]$  for the subgroups containing  $O_p(\mathcal{F})$ . We will show that  $\Xi$  is  $\mathcal{F}$ -stable, hence  $\Xi = \Lambda_{\mathcal{F}}$  and the result will follow.

So we must show for  $A \leq S$  and  $\varphi \in \mathcal{F}_{\text{iso}}(A, B)$  the following equalities:

$$|\Xi^{(A, \varphi)}| = |\Xi^{(A, \iota_A^S)}| = |\Xi^{(\varphi A, \varphi^{-1})}|.$$

If  $O_p(\mathcal{F}) \leq A$ , these equalities hold by construction of  $\Xi$ . Otherwise let  $\tilde{\varphi}_i$ ,  $i = 1, \dots, n$ , be the distinct extensions of  $\varphi$  to elements of  $\mathcal{F}_{\text{iso}}(O_p(\mathcal{F}) \cdot A, O_p(\mathcal{F}) \cdot B)$ . As in the proof of Proposition 9.10 we can write  $\Xi^{(A, \varphi)} = \coprod_{i=1}^n \Xi^{(PA, \tilde{\varphi}_i)}$ , so

$$|\Xi^{(A, \varphi)}| = \sum_{i=1}^n |\Xi^{(PA, \tilde{\varphi}_i)}| = n \cdot |\Xi^{(PA, \iota_{PA}^S)}|,$$

which depends only on the source  $A$  by Lemma 9.9. Therefore  $|\Xi^{(A,\varphi)}| = |\Xi^{(A,\iota_A^S)}|$ . Dually we can show that the fixed-point order depends only on the target of the isomorphism in question, so  $|\Xi^{(A,\iota_A^S)}| = |\Xi^{(\varphi A, \varphi^{-1})}|$ . This proves the result.  $\square$

Back on track: We haven't made use of the saturation of  $\mathcal{F}$  yet in this section; now we will need to in order to guarantee the existence of characteristic bisets for  $\mathcal{F}$ , in particular the unique minimal  $\mathcal{F}$ -characteristic biset  $\Lambda_{\mathcal{F}}$  for  $\mathcal{F}$ .

**Proposition 9.12.** *Let  $\Omega := \Lambda_{\mathcal{F}}$  be the minimal characteristic biset for  $\mathcal{F}$ , and let  $P \leq S$  fully  $K$ -normalized in  $\mathcal{F}$  for  $K \leq \text{Aut}(P)$ . If  $K \leq \text{Inn}(P)$  or  $K \geq \text{Inn}(P)$ , then  $N_{\Omega}^K(P)$  contains precisely one  $(N, N)$ -orbit isomorphic to  $[N, \text{id}_N]$ .*

*Proof.* We consider two cases.

(1)  $K = \{\text{id}_P\}$  or  $\text{Inn}(P) \leq K$ .

Fix  $\omega \in N_{\Omega}^K(P)$ ,  $\text{Stab}_{N \times N}(\omega) = (N, \text{id})$  and  $\text{Stab}_{S \times S}(\omega) = (Q, \psi)$ , so  $N \leq Q$  and  $\psi|_N = \text{id}_N$ . If  $K = \{\text{id}_P\}$ , the definition of  $C_{\Omega}(P) = N_{\Omega}^{\{\text{id}_P\}}(P)$  shows we must also have  $P \leq Q$  and  $\psi|_P = \text{id}_P$ . If  $\text{Inn}(P) \leq K$ , the definition of  $N_S^K(P)$  implies that  $P \cdot C_S(P) \leq N$ . In either case,  $P \cdot C_S(P) \leq Q$  and  $\psi|_{P \cdot C_S(P)} = \text{id}_{P \cdot C_S(P)}$ .

As  $P$  is fully  $K$ -normalized in  $\mathcal{F}$ , it is fully  $\mathcal{F}$ -centralized by Proposition 9.2, so [BLO, Proposition A.7] implies that  $P \cdot C_S(P)$  is  $\mathcal{F}$ -centric. As  $\text{id}_Q$  and  $\psi$  both restrict to the same automorphism of  $P \cdot C_S(P)$ , Proposition 8.3 says that there is some  $z \in Z(P \cdot C_S(P))$  such that  $\psi = \text{id}_Q \circ c_z|_Q = c_z|_Q$ . Since  $z \in S$ , the  $(S, S)$ -bisets  $[Q, c_z]$  and  $[Q, \iota_Q^S]$  are isomorphic.  $Q$  is  $\mathcal{F}$ -centric and  $\Omega$  is minimal, so Theorem 8.6 forces  $Q = S$ .

Thus all  $\omega \in N_{\Omega}^K(P)$  with  $\text{Stab}_{N \times N}(\omega) = (N, \text{id})$  live in the same  $(S, S)$ -orbit  $[S, \text{id}_S]$ , otherwise known as  $S$  with its natural  $(S, S)$ -biset structure. The subset of  ${}_S S_S$  that lies in  $N_{\Omega}(P)$  is  $N_S^K(P) = N$ , so all such points of  $N_{\Omega}^K(P)$  lie in the same  $(N, N)$ -orbit.

(2)  $K \leq \text{Inn}(P)$ .

Before dealing with the nonidentity subgroups of  $\text{Inn}(P)$ , we take a small detour to compare two different  $K$ -normalizers and their relation: Let  $K \leq \text{Aut}(P)$  be arbitrary with  $P$  fully  $K$ -normalized in  $\mathcal{F}$ , and set  $L := K \cdot \text{Inn}(P)$ . Note that  $\text{Inn}(P) \trianglelefteq \text{Aut}(P)$ , so  $L$  is in fact the product of  $K$  and  $\text{Inn}(P)$ , not merely the subgroup generated by the two. We have  $N_S^L(P) = P \cdot N_S^K(P)$ .

Now, consider the natural inclusion  $\iota: N_{\Omega}^K(P) \subseteq N_{\Omega}^L(P)$ . This is an  $(N_S^K(P), N_S^K(P))$ -equivariant map of bisets, hence  $\iota$  induces a map on orbits

$$\bar{\iota}: (N_S^K(P) \backslash N_{\Omega}^K(P) / N_S^K(P)) \rightarrow (N_S^L(P) \backslash N_{\Omega}^L(P) / N_S^L(P)).$$

We claim that  $\bar{\iota}$  is a bijection.

$\bar{\iota}$  is surjective: Suppose that  $\omega \in N_{\Omega}^L(P)$ ,  $\text{Stab}_{S \times S}(\omega) = (Q, \psi)$ . We have  $P \leq Q$ ,  $\psi P = P$ , and  $\psi|_P = \kappa \circ c_a$  for  $\kappa \in K$  and  $a \in P$ . By Remark 9.5, the point  $\omega \cdot a^{-1}$  has stabilizer  $({}^a Q, \psi \circ c_a^{-1})$  with  $P \leq {}^a Q$ ,  $(\psi \circ c_a^{-1})P = P$ , and  $(\psi \circ c_a^{-1})|_P = \kappa \circ c_a \circ c_a^{-1} = \kappa \in K$ . Thus  $\omega \cdot a^{-1} \in N_{\Omega}^K(P)$ , and as  $a \in P \leq N_S^L(P)$ , we see  $\bar{\iota}$  is surjective on orbits.

$\bar{\iota}$  is injective: Suppose that  $\omega_1, \omega_2 \in N_{\Omega}^K(P)$  have  $(S, S)$ -stabilizers  $(Q_i, \psi_i)$ ,  $i = 1, 2$ . We again have  $P \leq Q_i$ ,  $\psi_i P = P$ , and  $\psi_i|_P \in K$ . If  $\omega_1$  and  $\omega_2$  lie in the same  $(N_S^L(P), N_S^L(P))$ -orbit, there are elements  $a, b \in N_S^L(P)$  such that  $\omega_2 = a \cdot \omega_1 \cdot b$ . Since  $N_S^L(P) = P \cdot N_S^K(P)$  we may write  $b = p \cdot n$  for  $n \in N_S^K(P)$  and  $p \in P$ . As  $P \leq Q_1$ , we can write  $\omega_2 = a \cdot \omega_1 \cdot p \cdot n = (a\psi_1(p)) \cdot \omega_1 \cdot n$ . By Remark 9.5, the  $(S, S)$ -stabilizer of  $(a\psi_1(p)) \cdot \omega_1 \cdot n$  is  $((Q_1)^n, c_{a\psi_1(p)} \circ \psi_1 \circ c_n)$ . We already have  $(\psi_1 \circ c_n)|_P \in K$ , and the entire composite must restrict to an automorphism of  $P$  that lies in  $K$  because  $\omega_2 \in N_{\Omega}^K(P)$ . This forces

$c_{a\psi_1(p)}|_P \in K$ , or  $a\psi_1(p) \in N_S^K(P)$ . Thus  $\omega_1$  and  $\omega_2$  live in the same  $(N_S^K(P), N_S^K(P))$ -orbit, and injectivity is proved.

In fact, we have shown more: Given any subgroup  $H \leq \text{Aut}(P)$  of automorphisms such that  $H \leq K \leq L := H \cdot \text{Inn}(P) = K \cdot \text{Inn}(P)$ , we have that the inclusions  $N_\Omega^H(P) \subseteq N_\Omega^L(P)$  and  $N_\Omega^K(P) \subseteq N_\Omega^L(P)$  both induce bijections on orbits, so in fact the third natural inclusion  $N_\Omega^H(P) \subseteq N_\Omega^K(P)$  must induce a bijection on orbits as well.

In particular, consider the case that  $H = \{\text{id}_P\}$ ,  $L = \text{Inn}(P)$ , and  $K \leq \text{Inn}(P)$  is arbitrary. Then  $N_S^H(P) = C_S(P)$ , and we've already seen that there is a unique  $(C_S(P), C_S(P))$ -orbit of  $C_\Omega(P)$  with stabilizer  $(C_S(P), \text{id}_{C_S(P)})$ . There is some  $\omega \in \Omega$  that has  $(S, S)$ -stabilizer  $(S, \text{id})$ , so  $\omega \in C_\Omega(P) \subseteq N_\Omega^K(P)$  and has  $(N, N)$ -stabilizer  $(N, \text{id}_N)$  as an element of  $N_\Omega^K(P)$ . Suppose that there is some other  $\omega' \in N_\Omega^K(P)$  with  $(N, N)$ -stabilizer  $(N, \text{id}_N)$ . Then  $\omega' \in C_\Omega(P)$  and has  $(C_S(P), C_S(P))$ -stabilizer  $(C_S(P), \text{id}_{C_S(P)})$ , and as we have already proved our result for the centralizer biset, we conclude that  $\omega$  and  $\omega'$  must lie in the same  $(C_S(P), C_S(P))$ -orbit, and hence in the same  $(N, N)$ -orbit as well. This proves the result for arbitrary subgroups of  $\text{Inn}(P)$ .  $\square$

In the course of the proof of Proposition 9.12 we made use of the following interesting fact, which we record here for ease of reference:

**Proposition 9.13.** *Let  $\Omega$  be a semicharacteristic biset for  $\mathcal{F}$ ,  $P$  a subgroup of  $S$ , and  $H, K \leq \text{Aut}(P)$  two groups of automorphisms satisfying*

$$H \leq K \leq H \cdot \text{Inn}(P) = K \cdot \text{Inn}(P).$$

*Then the natural inclusion  $N_\Omega^H(P) \subseteq N_\Omega^K(P)$  induces a bijection on orbits:*

$$(N_S^H(P) \backslash N_\Omega^H(P) / N_S^H(P)) \cong (N_S^K(P) \backslash N_\Omega^K(P) / N_S^K(P)).$$

*Remark 9.14.* We could use Propositions 9.12 and 9.13 to reprove Puig's main theorem on  $K$ -normalizers (cf. [Pui2, Proposition 21.11]): If  $\Omega$  is a characteristic biset for  $\mathcal{F}$  with  $P \leq S$  and  $K \leq \text{Aut}(P)$  given so that  $P$  is fully  $K$ -normalized in  $\mathcal{F}$ , then  $N_\Omega^K(P)$  is a characteristic biset for  $N_{\mathcal{F}}^K(P)$ .

[Sketch of proof:  $N_\Omega^K(P)$  is always  $\mathcal{N}$ -semicharacteristic by Proposition 9.10, so we only need show that  $p \nmid |N_\Omega^K(P)|/|N_S^K(P)|$ . In the case that  $K$  contains or is contained in  $\text{Inn}(P)$ , this is a direct calculation based on Proposition 9.12; in the general case one can use Proposition 9.13 to show  $|N_\Omega^K(P)|/|N_S^K(P)| = |N_\Omega^L(P)|/|N_S^L(P)|$  where  $L := K \cdot \text{Inn}(P)$ , and that  $P$ 's being fully  $K$ -normalized in  $\mathcal{F}$  implies that it is also fully  $L$ -normalized. From this the result follows.]

In particular, the existence of a  $N_{\mathcal{F}}^K(P)$ -characteristic biset implies that  $N_{\mathcal{F}}^K(P)$  is a saturated fusion system. There is little gained by reproving this result in detail; instead we will assume it and derive the following more precise formulation.

**Theorem 9.15.** *Suppose that  $\Omega = \Lambda_{\mathcal{F}}$  is the minimal characteristic biset for  $\mathcal{F}$ . Suppose  $P \leq S$  and  $K \leq \text{Aut}(S)$  such that  $K$  either contains or is contained in  $\text{Inn}(P)$ .*

*If  $P$  is fully  $K$ -normalized in  $\mathcal{F}$ , then  $N_\Omega^K(P)$  is a characteristic  $(N, N)$ -biset for  $\mathcal{N} = N_{\mathcal{F}}^K(P)$  that contains precisely one copy of  $\Lambda_{\mathcal{N}}$ , the minimal characteristic biset for  $\mathcal{N}$ .*

*Moreover, if  $P$  is  $\mathcal{F}$ -centric, then  $N_\Omega^K(P) = \Lambda_{\mathcal{N}}$ .*

*Proof.* By [Pui2, Proposition 21.11],  $\mathcal{N}$  is a saturated fusion system on  $N$ , hence our parameterization of semicharacteristic bisets applies.  $N_\Omega^K(P)$  is  $\mathcal{N}$ -semicharacteristic by Proposition 9.10, and by Theorem 5.3 the number of copies of  $\Lambda_{\mathcal{N}}$  contained in  $N_\Omega^K(P)$



is equal to the number of orbits isomorphic to  $[N, \text{id}_N]$ . By Proposition 9.12, there is a unique such  $(N, N)$ -orbit, proving the first statement.

Now, suppose that  $P$  is  $\mathcal{F}$ -centric. To show that  $N_\Omega^K(P) = \Lambda_{\mathcal{N}}$ , it suffices to show that there are no other minimal  $\mathcal{N}$ -semicharacteristic bisets beyond  $\Lambda_{\mathcal{N}}$  contained in  $N_\Omega^K(P)$ . Suppose that  $\omega \in N_\Omega^K(P)$  has  $(N, N)$ -stabilizer  $(A, \iota_A^N)$ , then  $P \leq A \leq N$  by Proposition 9.11. Then the  $(S, S)$ -stabilizer of  $\omega$  is  $(Q, \psi)$ , with  $A = Q \cap N$  and  $\psi|_A = \text{id}_A$ . All groups in sight are  $\mathcal{F}$ -centric by assumption that  $P$  is, so we may use Theorem 8.3 to conclude that  $\psi = c_z|_Q$  for some  $z \in Z(A)$ . Therefore  $[Q, \psi] = [Q, c_z] = [Q, \iota_Q^S]$ , and we know from Theorem 8.6 that the only such orbit in  $\Lambda_{\mathcal{F}}$  when  $Q$  is  $\mathcal{F}$ -centric is  $[S, \text{id}_S]$ . We conclude that  $Q = S$  and  $N \leq Q$ . Therefore the only point-stabilizer of  $N_\Omega^K(P)$  of the form  $(A, \iota_A^N)$  is  $(N, \text{id}_N)$ , so the only  $\mathcal{N}$ -semicharacteristic bisets contained in  $N_\Omega^K(P)$  are copies of  $\Lambda_{\mathcal{N}}$ . As we have seen that there is exactly one of these, we have  $N_\Omega^K(P) = \Lambda_{\mathcal{N}}$ , as claimed.  $\square$

**Conjecture 9.16.**  *$P$  need not be  $\mathcal{F}$ -centric for the conclusions of Theorem 9.15 to hold: If  $\Omega = \Lambda_{\mathcal{F}}$  is the minimal characteristic biset for  $\mathcal{F}$  and we are given  $P \leq S$  and  $K \leq \text{Aut}(P)$  such that  $K$  contains or is contained in  $\text{Inn}(P)$ , and if  $P$  is fully  $K$ -normalized in  $\mathcal{F}$ , then  $N_\Omega^K(P) = \Lambda_{\mathcal{N}}$ , the minimal characteristic biset for  $\mathcal{N}$ .*

*Counterexample 9.17.* There can be no analogue of Conjecture 9.16 that completely relaxes the conditions on  $K \leq \text{Aut}(P)$  in Proposition 9.12 and Theorem 9.15 and still have the conclusions hold:

Let  $\mathbb{Z}/3$  act on  $Q_8$  by permuting the elements  $i, j, k$  cyclically. Set  $G := Q_3 \rtimes \mathbb{Z}/3$ ,  $\mathcal{F} := \mathcal{F}_{Q_8}(G)$ , and  $K = \mathbb{Z}/3 \leq \text{Aut}(Q_8)$ . Note that  $Q_8$  is fully  $K$ -normalized in  $\mathcal{F}$ . Since  $K$  is a  $2'$ -group, we have  $N_{Q_8}^K(Q_8) = Z(Q_8) \cong \mathbb{Z}/2$ . If  $\kappa$  is a generator for  $K$ , one easily checks that the minimal  $\mathcal{F}$ -characteristic biset is  $\Lambda_{\mathcal{F}} = [Q_8, \text{id}_{Q_8}] \amalg [Q_8, \kappa] \amalg [Q_8, \kappa^2]$ . One can further calculate that  $N_{\Lambda_{\mathcal{F}}}^K(Q_8) = 3 \cdot [Z(Q_8), \text{id}_{Z(Q_8)}]$ , contrary to the conclusion of Proposition 9.12.

*Remark 9.18.* Counterexample 9.17 shows in particular that there must be some condition imposed on  $K \leq \text{Aut}(P)$  in general to guarantee that  $N_{\Lambda_{\mathcal{F}}}^K(P) = \Lambda_{N_{\mathcal{F}}(P)}$ . We have seen that it is enough (when  $P$  is  $\mathcal{F}$ -centric) to assume that  $K$  either contains or is contained in  $\text{Inn}(P)$ . While it is possible that one could find a larger class of subgroups of  $\text{Aut}(P)$  for which the conclusion of Theorem 9.15 holds, we have at least already covered the most important examples with our current formulation: If  $K = \{\text{id}\}$  or  $K = \text{Aut}(P)$  we get the minimal characteristic bisets  $C_\Omega(P)$  and  $N_\Omega(P)$  for the fusion systems  $C_{\mathcal{F}}(P)$  and  $N_{\mathcal{F}}(P)$ , respectively. We also cover the cases of the subsystems  $Q \cdot C_{\mathcal{F}}(Q)$  (on  $Q \cdot C_S(P)$ ) and  $N_P(Q) \cdot C_{\mathcal{F}}(Q)$  (on  $N_S(P)$ ) corresponding to the cases  $K = \text{Inn}(P)$  and  $K = \text{Aut}_S(P)$ , respectively (cf. [Lin, Definition 3.1]).

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*E-mail address:* mgelvin@gmail.com

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, COPENHAGEN, DENMARK

*E-mail address:* spr@math.ku.dk